



International
Scientific Conference



Algebraic and Geometric Methods of Analysis



Devoted to 160 anniversary of
Dvytro Grave
(25.08.1863 - 19.12.1939)
Academician of the Ukrainian
Academy of Sciences, the
first director of the Institute of
Mathematics of NAS of Ukraine

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LIST OF TOPICS

- Algebraic methods in geometry
- Differential geometry in the large
- Geometry and topology of differentiable manifolds
- General and algebraic topology
- Dynamical systems and their applications
- Geometric and topological methods in natural sciences
- Geometric problems in mathematical analysis

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Hölder Continuity of Generalized Harmonic Functions in the Unit Disc

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The main purpose of this talk is to discuss about the membership in Hölder classes for (p, q) -harmonic functions $u = K_{p,q}[f]$ such that their boundaries functions $f \in \Lambda_\beta(\mathbb{T})$.

Consider the second order partial differential operators, studied in [2], of the form

$$L_{p,q} := (1 - |z|^2)\partial\bar{\partial} + pz\partial + q\bar{z}\bar{\partial} - pq, \quad z \in \mathbb{D}, \quad (1)$$

where p, q are real parameters. We say that a function u is (p, q) -harmonic if u is twice continuously differentiable in $\mathbb{D}$ and $L_{p,q}u = 0$.

Let consider the associated Dirichlet boundary value problem of functions u , satisfying the equation $L_{p,q}u = 0$,

$$\begin{aligned} L_{p,q}u &= 0 \quad \text{in } \mathbb{D}, \\ u &= f \quad \text{on } \mathbb{T}. \end{aligned} \quad (2)$$

For $p, q \in \mathbb{R} \setminus \mathbb{Z}^-$ such that $p + q > -1$, the (p, q) -harmonic Poisson kernel is defined by

$$K_{p,q}(z) = c_{p,q} \frac{(1 - |z|^2)^{p+q+1}}{(1 - z)^{p+1}(1 - \bar{z})^{q+1}}, \quad c_{p,q} = \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+1)},$$

where Γ is the Gamma function.

The Solution of (2) has the following (p, q) -harmonic integral representation of $f \in L^1(\mathbb{T})$ which defined by

$$u(z) = K_{p,q}[f](z) := \frac{1}{2\pi} \int_0^{2\pi} K_{p,q}(ze^{-i\theta})f(e^{i\theta})d\theta, \quad z \in \mathbb{D}. \quad (3)$$

Remark that if $p = q = 0$, then the solution u is the classical harmonic function.

Let us recall the notion of "Hölder" continuity

Definition 1. For a bounded subset E of the complex plane, let ω be a *majorant*, i.e., a continuous increasing function on $[0, \infty)$ such that $\omega(0) = 0$ and $\omega(t)/t$ is non-increasing for $t > 0$. For a real or complex valued function f on E we write $f \in \Lambda_\omega(E)$ if there is a constant $C > 0$ such that

$$|f(z_1) - f(z_2)| \leq C\omega(|z_1 - z_2|), \quad z_1, z_2 \in E.$$

If $\omega(t) = t^\beta$, the class is simply denoted by $\Lambda_\beta(E)$ which is commonly referred to as the Hölder class for the set E of order $\beta \in (0, 1]$.

The following growth estimate is useful.

Lemma 2. [3] Let $u \in \mathcal{C}^1(\mathbb{D})$ and ω be a majorant satisfying the Dini condition, that is,

$$\tilde{\omega}(x) := \int_0^x \frac{\omega(t)}{t} dt < \infty, \quad x > 0.$$

If f satisfies $|\partial u(z)| + |\bar{\partial} u(z)| \leq C \frac{\omega(1-|z|^2)}{1-|z|^2}$ for all $z \in \mathbb{D}$, then $u \in \Lambda_{\tilde{\omega}}(\mathbb{D})$.

As a consequence of the previous result we get the first main result

Theorem 3. [1] Let $p + q > 1$ and $0 < \beta \leq 1$. Let $f \in \Lambda_{\beta}(\mathbb{T})$ and set $u = K_{p,q}[f]$. It yields

- (1) If $p + q \neq \beta - 1$, then $u \in \Lambda_{\min\{\beta, p+q+1\}}(\mathbb{D})$.
- (2) If $p + q = \beta - 1$, then $u \in \Lambda_{\omega_{\beta}}(\mathbb{D})$, where $\omega_{\beta}(t) := t^{\beta} (1 - \log(t))$.

In particular $u \in \bigcap_{0 < \alpha < \beta} \Lambda_{\alpha}(\mathbb{D})$.

In particular, for $\beta = 1$, and $-1 < p + q < 0$, we have $u \in \Lambda_{p+q+1}(\mathbb{D})$ and we provide an example where $u \notin \Lambda_1(\mathbb{D})$. This example shows the failure of the stability of Lipschitz continuity in the case $p + q < 0$, i.e we provide new examples of functions $f \in \Lambda_1(\mathbb{T})$ such that $u = K_{p,q}[f] \notin \Lambda_1(\mathbb{D})$, when $p + q \in (-1, 0]$. For more details we refer the reader to [1].

Example 4. [1] Let consider two cases:

- The case $-1 < p + q < 0$: let $k \in \mathbb{Z}$,

$$u_k(z) := \frac{1}{2\pi} \int_0^{2\pi} K_{p,q}(ze^{-i\theta}) e^{ik\theta} d\theta, \quad z \in \mathbb{D}.$$

- The case $p + q = 0$ and $p \neq 0$: let

$$u_0(z) := \frac{1}{2\pi} \int_0^{2\pi} K_{p,q}(ze^{-i\theta}) d\theta = c_{p,q} F(-p, -q; 1; |z|^2), \quad z \in \mathbb{D}.$$

where F is the Gaussian hypergeometric function. Through these two case we can prove that one of the partial derivative of u_k (resp. u_0) is not bounded, which leads to $u_k \notin \Lambda_1(\mathbb{D})$ (resp. $u_0 \notin \Lambda_1(\mathbb{D})$).

Definition 5. A sense-preserving diffeomorphism u is said to be K -quasiconformal, if

$$\frac{|\partial u(z)| + |\bar{\partial} u(z)|}{|\partial u(z)| - |\bar{\partial} u(z)|} \leq K,$$

throughout the given region Ω , where $K \in [1, \infty)$ is a constant.

Under an extra condition, we preserve the same order of f and u .

Theorem 6. [1] Let $p + q > 1$ and $0 < \beta \leq 1$, $f \in \Lambda_{\beta}(\mathbb{T})$ and set $u = K_{p,q}[f]$, be a K -quasiconformal mapping. Then $u \in \Lambda_{\beta}(\mathbb{D})$.

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E. Lytvynov <i>Lie structures of the Sheffer group over a Hilbert space</i>	58
R. El Maaouy, D. Bennis, L. Oyonarte, J. R. G. Rozas <i>The Gorenstein flat model structure relative to a semidualizing module</i>	60
O. Makarchuk <i>On the structure of the distribution of one random series.</i>	61
S. Maksymneko <i>Homotopy types of diffeomorphisms groups of simplest Morse-Bott foliations on lens spaces</i>	62
Iu. Marko <i>Spaces of idempotent measures with countable support</i>	62
S. Marouaniv <i>SKT hyperbolic and Gauduchon hyperbolic compact complex manifolds</i>	63
N. Mazurenko, M. Zarichnyi <i>Invariant $*$-measures</i>	66
M. Mhamdi <i>Hölder Continuity of Generalized Harmonic Functions in the Unit Disc</i>	67
Ł. Michalak <i>Reeb graph invariants of Morse functions, manifolds and groups</i>	69
P. Mormul <i>Car+trailers' systems are locally nilpotentizable (a Trieste 2000 conference revisited)</i>	70
J. Morris <i>Degree theory for proper C^1 Fredholm mappings with applications to boundary value problems on the half line</i>	70
S. Myroshnychenko, K. Tatarko, V. Yaskin <i>How far apart can the projection of the centroid of a convex body and the centroid of its projection be?</i>	71
M. Nesterenko <i>Contractions of representations and realizations of Lie algebras</i>	73
Yu. Nikolayevsky <i>Geodesic orbit pseudo Riemannian nilmanifolds</i>	74
Z. Novosad, A. Zagorodnyuk <i>The conditions of hypercyclicity of weighted backward shifts</i>	75
T. Obikhod <i>Studying the properties of a superpotential using algebraic equations</i>	76
P. O. Olanipekun <i>On critical submanifolds of the Willmore energy in four dimensions</i>	78
I. Ovtsynov <i>Fermat–Torricelli sets of finite sets of points in Euclidean plane</i>	80
C. A. Pallikaros <i>Degenerations of complex associative algebras of dimension three</i>	82
J. F. Peters, F. Peu, J. Zia <i>Several forms of the geometric Lusternik–Schnirel'mann category</i>	82