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- Algebraic methods in geometry
- Differential geometry in the large
- Geometry and topology of differentiable manifolds
- General and algebraic topology
- Dynamical systems and their applications
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Centrally extended generalization of the superconformal loop Lie algebra and integrable heavenly type systems on supermanifolds

Oksana Ye. Hentosh
(Pidstryhach Inst. for Applied Problems of Mech. and Math., NASU, Lviv, Ukraine)
E-mail: ohen@ua.fm

Let us consider the semi-direct sum $\tilde{\mathcal{G}} \ltimes \tilde{\mathcal{G}}^*_{reg}$ of the loop Lie algebra $\tilde{\mathcal{G}} := \widetilde{diff}(\mathbb{T}^{1|N})$, consisting of the superconformal vector fields on a supertor $\mathbb{T}^{1|N}$ in the forms:

$$\tilde{a} := a\partial/\partial x + \frac{1}{2}\sum\nolimits_{i=1}^{N}(D_{\vartheta_i}a)D_{\vartheta_i}, \quad a := a(x,\vartheta;\lambda), \tag{1}$$

where $a \in C^\infty(\mathbb{T}^{1|N} \times (\mathbb{D}^1_+ \cup \mathbb{D}^1_-);\Lambda_0)$, $(x,\vartheta) \in \mathbb{T}^{1|N} \simeq \mathbb{S}^1 \times \Lambda_1^N$, $\Lambda := \Lambda_0 \oplus \Lambda_1$ is a infinite-dimensional Grassmann algebra over $\mathbb{C} \supset \Lambda_0$, $\vartheta := (\vartheta_1, \vartheta_2, \ldots, \vartheta_N)$ and $D_{\vartheta_i} := \partial/\partial\vartheta_i + \vartheta_i\partial/\partial x$, $i = \overline{1,N}$, which are holomorphic in the "spectral" parameter $\lambda \in \mathbb{C}$ on the interior $\mathbb{D}^1_+ \subset \mathbb{C}$ and exterior $\mathbb{D}^1_- \subset \mathbb{C}$ regions of the unit centrally located disk $\mathbb{D}^1 \subset \mathbb{C}$, and its regular dual space $\tilde{\mathcal{G}}^*_{reg}$ with respect to the parity:

$$(\tilde{l},\tilde{a})_0 = \operatorname{res}\lambda^{-1}\int_{\mathbb{T}^{1|N}} dx d^N\vartheta\,(la), \quad \tilde{l} := l(x,\vartheta;\lambda)(dx + \sum\nolimits_{i=1}^{N}\vartheta_i d\vartheta_i) \in \tilde{\mathcal{G}}^*_{reg}, \tag{2}$$

where $l \in C^\infty(\mathbb{T}^{1|(2k-1)} \times (\mathbb{D}^1_+ \cup \mathbb{D}^1_-);\Lambda_1)$ if $N = 2k-1$ and $l \in C^\infty(\mathbb{T}^{1|2k} \times (\mathbb{D}^1_+ \cup \mathbb{D}^1_-);\Lambda_0)$ if $N = 2k$, $k \in \mathbb{N}$. The superconformal loop Lie algebra $\tilde{\mathcal{G}}$ possesses the commutator:

$$[\tilde{a},\tilde{b}] = \tilde{c}, \quad \tilde{c} := c\partial/\partial x + \frac{1}{2}\sum\nolimits_{i=1}^{N}(D_{\vartheta_i}c)D_{\vartheta_i},$$
$$c := a(\partial b/\partial x) - b(\partial a/\partial x) + \frac{1}{2}\sum\nolimits_{i=1}^{N}(D_{\vartheta_i}a)(D_{\vartheta_i}b), \quad \tilde{a},\tilde{b} \in \tilde{\mathcal{G}},$$

splits into the direct sum of its Lie subalgebras $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$, for which the following dual spaces are identified: $\tilde{\mathcal{G}}^*_{+,reg} \simeq \tilde{\mathcal{G}}_-$, $\tilde{\mathcal{G}}^*_{-,reg} \simeq \tilde{\mathcal{G}}_+$. Here $\tilde{a}(\infty) = 0$ for any $\tilde{a}(\lambda) \in \tilde{\mathcal{G}}_-$. On $\tilde{\mathcal{G}} \ltimes \tilde{\mathcal{G}}^*_{reg}$ one determines the commutator:

$$[\tilde{a} \ltimes \tilde{l}, \tilde{b} \ltimes \tilde{m}] := [\tilde{a},\tilde{b}] \ltimes (ad^*_{\tilde{a}}\tilde{m} - ad^*_{\tilde{b}}\tilde{l}), \quad \tilde{a},\tilde{b} \in \tilde{\mathcal{G}}, \quad \tilde{l},\tilde{m} \in \tilde{\mathcal{G}}^*_{reg},$$

where ad^* is the co-adjoint action of $\tilde{\mathcal{G}}$ with respect to the parity (2) and

$$ad^*_a l = l_x a + \frac{4-N}{2} l a_x + \frac{(-1)^{N+1}}{2}\sum\nolimits_{i=1}^{N}(D_{\vartheta_i}l)(D_{\vartheta_i}a)$$

for any vector field $\tilde{a} \in \tilde{\mathcal{G}}$ and a fixed element $\tilde{l} \in \tilde{\mathcal{G}}^*_{reg}$, as well as nondegenerate symmetric bilinear form:

$$(\tilde{a} \ltimes \tilde{l}, \tilde{b} \ltimes m) = (\tilde{l},\tilde{b})_0 + (\tilde{m},\tilde{a})_0.$$

One constructs the central extension $\hat{\tilde{\mathfrak{G}}} := \tilde{\mathfrak{G}} \oplus \mathbb{C}$ of the Lie algebra $\tilde{\mathfrak{G}} := \prod_{z\in\mathbb{S}^1}(\tilde{\mathcal{G}} \ltimes \tilde{\mathcal{G}}^*_{reg})$ by the 2-cocycle [1]:

$$\omega_2(\tilde{a} \ltimes \tilde{l}, \tilde{b} \ltimes m) = \int_{\mathbb{S}^1} dz((\tilde{l},\partial\tilde{b}/\partial z)_0 - (\tilde{m},\partial\tilde{a}/\partial z)_0), \quad (\tilde{a} \ltimes \tilde{l}), (\tilde{b} \ltimes \tilde{m}) \in \tilde{\mathfrak{G}}, \quad z \in \mathbb{S}^1.$$

The Lie algebra $\tilde{\mathfrak{G}}$ permits the standard splitting $\tilde{\mathfrak{G}} := \tilde{\mathfrak{G}}_+ \oplus \tilde{\mathfrak{G}}_-$ of the Lie algebra $\tilde{\mathfrak{G}}$ into the direct sum of its Lie subalgebras $\tilde{\mathfrak{G}}_+ := \prod_{z\in\mathbb{S}^1}(\tilde{\mathcal{G}}_+ \ltimes \tilde{\mathcal{G}}^*_{-,reg})$ and $\tilde{\mathfrak{G}}_- := \prod_{z\in\mathbb{S}^1}(\tilde{\mathcal{G}}_- \ltimes \tilde{\mathcal{G}}^*_{+,reg}))$. Thus, by means of the $\mathcal{R}$-operator approach [2] one introduces the following Lie-Poisson bracket:

$$\{\mu,\nu\}_{\mathcal{R}} = (\tilde{a} \ltimes \tilde{l}, [R\nabla_r\mu(\tilde{a} \ltimes \tilde{l}), \nabla_l\nu(\tilde{a} \ltimes \tilde{l})] + [\nabla_r\mu(\tilde{a} \ltimes \tilde{l}), R\nabla_l\nu(\tilde{a} \ltimes \tilde{l})]) +$$
$$+\omega_2(R\nabla_r\mu(\tilde{a} \ltimes \tilde{l}), \nabla_l\nu(\tilde{a} \ltimes \tilde{l})) + \omega_2(\nabla_r\mu(\tilde{a} \ltimes \tilde{l}), R\nabla_l\nu(\tilde{a} \ltimes \tilde{l})), \tag{3}$$

where $\mu, \nu \in \mathcal{D}(\tilde{\mathfrak{G}}^*)$ are arbitrary smooth by Frechet functionals on $\tilde{\mathfrak{G}}^*$, $\mathcal{R} = (P_+ - P_-)/2$, P_+ and P_- are projectors on $\tilde{\mathfrak{G}}_+$ and $\tilde{\mathfrak{G}}_-$ respectively, on the dual space $\tilde{\mathfrak{G}}^* \simeq \tilde{\mathfrak{G}}$ to the Lie algebra $\tilde{\mathfrak{G}}$. Here $\nabla_l h(\tilde{a} \ltimes \tilde{l}) := (\nabla_l h_{\tilde{l}} \ltimes \nabla_l h_{\tilde{a}}) \in \tilde{\mathfrak{G}}$ and $\nabla_r h(\tilde{a} \ltimes \tilde{l}) := (\nabla_r h_{\tilde{l}} \ltimes \nabla_r h_{\tilde{a}}) \in \tilde{\mathfrak{G}}$ are left and right gradients of any smooth functional $h \in \mathcal{D}(\tilde{\mathfrak{G}}^*)$ at a point $(\tilde{a} \ltimes \tilde{l}) \in \tilde{\mathfrak{G}}^*$. Due to the Adler-Kostant-Symes theory [2] the Lie-Poisson bracket (3) generates the hierarchy of Hamiltonian flows:

$$\partial(\tilde{a} \ltimes \tilde{l})/\partial t_p := -ad^*_{P_+\nabla_l h^{(p)}(\tilde{a}\ltimes\tilde{l})}(\tilde{a} \ltimes \tilde{l}) = \{\tilde{a} \ltimes \tilde{l}, h^{(p)}\}_{\mathcal{R}}, \quad (\tilde{a} \ltimes \tilde{l}) \in \tilde{\mathfrak{G}}^*, \quad p \in \mathbb{Z}_+,$$

where $P_+\nabla_l h^{(p)}(\tilde{a} \ltimes \tilde{l}) = (\nabla_l h^{(p)}_{\tilde{l},+} \ltimes \nabla_l h^{(p)}_{\tilde{a},+})$, $\nabla_l h^{(p)}(\tilde{a} \ltimes \tilde{l}) = \lambda^p \nabla_l h(\tilde{a} \ltimes \tilde{l})$, $\nabla_l h_{\tilde{l}} \sim \sum_{j\in\mathbb{Z}_+} \nabla_l h_{\tilde{l},j}\lambda^{-j}$ and $\nabla_l h_{\tilde{a}} \sim \sum_{j\in\mathbb{Z}_+} \nabla_l h_{\tilde{a},j}\lambda^{-j}$ as $|\lambda| \to \infty$, for any Casimir invariant $h \in I(\hat{\mathfrak{G}}^*)$, satisfying, by definition, the following relationship:

$$ad^*_{\nabla_l h(\tilde{a}\ltimes\tilde{l})}(\tilde{a} \ltimes \tilde{l}) = 0, \quad (\tilde{a} \ltimes \tilde{l}) \in \tilde{\mathfrak{G}}^*.$$

Any two Hamiltonian flows on $\tilde{\mathfrak{G}}^*$ in the forms:

$$\partial(\tilde{a} \ltimes \tilde{l})/\partial y = \{\tilde{a} \ltimes \tilde{l}, h^{(y)}(\tilde{a} \ltimes \tilde{l})\}_{\mathcal{R}}, \quad \partial(\tilde{a} \ltimes \tilde{l})/\partial t = \{\tilde{a} \ltimes \tilde{l}, h^{(t)}(\tilde{a} \ltimes \tilde{l})\}_{\mathcal{R}},$$

where $\nabla_l h^{(y)} = \lambda^{p_y}\nabla_l h(\tilde{a} \ltimes \tilde{l})$, $\nabla_l h^{(t)} = \lambda^{p_t}\nabla_l h(\tilde{a} \ltimes \tilde{l})$, $p_y, p_t \in \mathbb{Z}_+$, and $h \in I(\hat{\mathfrak{G}}^*)$, give rise to the separately commuting evolution equations:

$$\partial\tilde{a}/\partial y = -[\nabla_l h^{(y)}_{\tilde{l},+}, \tilde{a}] + \partial(\nabla_l h^{(y)}_{\tilde{l},+})/\partial z, \quad \partial\tilde{a}/\partial t = -[\nabla_l h^{(t)}_{\tilde{l},+}, \tilde{a}] + \partial(\nabla_l h^{(t)}_{\tilde{l},+})/\partial z, \tag{4}$$

and

$$\partial\tilde{l}/\partial y = -ad^*_{\nabla_l h^{(y)}_{\tilde{l},+}}\tilde{l} + ad^*_{\tilde{a}}\nabla_l h^{(y)}_{\tilde{a},+} + \partial(\nabla_l h^{(y)}_{\tilde{a},+})/\partial z,$$
$$\partial\tilde{l}/\partial t = -ad^*_{\nabla_l h^{(t)}_{\tilde{l},+}}\tilde{l} + ad^*_{\tilde{a}}\nabla_l h^{(t)}_{\tilde{a},+} + \partial(\nabla_l h^{(t)}_{\tilde{a},+})/\partial z.$$

Proposition 1. *The commutativity of evolutions (4) is equivalent to the relationship:*

$$[\nabla_l h^{(y)}_{\tilde{l},+}, \nabla_l h^{(t)}_{\tilde{l},+}] - \partial(\nabla_l h^{(y)}_{\tilde{l},+})/\partial t + \partial(\nabla_l h^{(t)}_{\tilde{l},+})/\partial y = 0, \tag{5}$$

which is reduced on every coadjoint orbit of the Lie algebra $\hat{\mathfrak{G}}$ to the Lax-Sato representation for some system of nonlinear heavenly type equations on a functional supermanifold. The relationship (5) is a compatibility condition for the following linear vector equations:

$$\partial\psi/\partial y + \nabla_l h^{(y)}_{\tilde{l},+}\psi = 0, \quad \partial\psi/\partial z + \tilde{a}\psi = 0, \quad \partial\psi/\partial t + \nabla_l h^{(t)}_{\tilde{l},+}\psi = 0,$$

where $(y, t; \lambda, z, x, \theta) \in (\mathbb{R}^2 \times (\mathbb{C} \times \mathbb{S}^1 \times \mathbb{T}^{1|N}))$ and $\psi \in C^2(\mathbb{R}^2 \times (\mathbb{C} \times \mathbb{S}^1 \times \mathbb{T}^{1|N}); \mathbb{C})$.

By use of the Lax-Sato compatibility condition (5) one can construct integrable systems of heavenly type equations on functional supermanifolds, which can be considered as generalizations of Lax-Sato integrable superanalogs [3] of the Mikhalev-Pavlov heavenly type equation, choosing the smooth functions $a := \sum_{k=1}^{K-1} w_{k,x}(x, \theta)\lambda^k - \lambda^K$ and $l := \sum_{k=1}^{K-1} \xi_{k,x}(x, \theta)\lambda^k$, $K \in \mathbb{N}$, in (1) and (2) respectively.

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Зміст