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# Extension theorems for holomorphic bundles on complex manifolds with boundary

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We begin with the following important result due to Donaldson [Do] for Kähler, and Xi [Xi] for general Hermitian complex manifolds with boundary:

**Theorem 1.** *Let  $\bar{X}$  be a compact complex manifold with non-empty boundary  $\partial\bar{X}$ ,  $g$  be a Hermitian metric on  $\bar{X}$  and  $\mathcal{E}$  be a holomorphic bundle on  $\bar{X}$ . Let  $h$  be a Hermitian metric on the restriction  $\mathcal{E}|_{\partial X}$ . There exists a unique Hermitian metric  $H$  on  $\mathcal{E}$  satisfying the conditions*

$$\Lambda_g F_H = 0, \quad H|_{\partial X} = h,$$

where  $F_H \in A^2(\bar{X}, \text{End}(\mathcal{E}))$  denotes the curvature of the Chern connection associated with  $H$ .

Note that the map  $H \mapsto \Lambda_g F_H$  is a non-linear second order elliptic differential operator, so the system  $\Lambda_g F_H = 0$ ,  $H|_{\partial\bar{X}} = h$  can be viewed as a non-linear Dirichlet problem. The theorem of Donaldson and Xi states that this non-linear Dirichlet problem is always uniquely solvable.

Note also that the analogue statement for closed manifolds (i.e. in the case  $\partial\bar{X} = \emptyset$ ) does not hold. Indeed, the classical Kobayashi-Hitchin correspondence states that, for a holomorphic bundle  $\mathcal{E}$  on a closed Hermitian manifold  $(X, g)$ , the equation  $\Lambda_g F_H = 0$  is solvable if and only if  $\deg_g(\mathcal{E}) = 0$  (which is a topological condition if  $g$  is Kählerian) and  $\mathcal{E}$  is polystable with respect to  $g$  (see [LT]).

Recall that a unitary connection  $\nabla$  on a Hermitian differentiable bundle  $(E, H)$  on  $\bar{X}$  is called Hermitian Yang-Mills if  $\Lambda_g F_\nabla = 0$ ,  $F_\nabla^{0,2} = 0$ . In the classical case  $\dim_{\mathbb{C}}(X) = 2$  – which plays a fundamental role in Donaldson theory – these conditions are equivalent to the anti-self-duality condition  $F_\nabla^+ = 0$ .

In [Do] Donaldson shows that Theorem 1 has important geometric consequences:

**Corollary 2.** *Let  $\bar{X}$  be a compact complex manifold with non-empty boundary,  $g$  be a Hermitian metric on  $\bar{X}$  and  $(E, H)$  be a Hermitian differentiable bundle on  $\bar{X}$ . There exists a natural bijection between:*

- (1) *the moduli space of pairs  $(\mathcal{E}, \theta)$  consisting of a holomorphic structure  $\mathcal{E}$  on  $E$  and a differentiable trivialization  $\theta$  of  $E|_{\partial\bar{X}}$ ,*
- (2) *the moduli space of pairs  $(\nabla, \tau)$  consisting of a Hermitian Yang-Mills connection on  $(E, H)$  and a differentiable unitary trivialization  $\tau$  of  $E|_{\partial\bar{X}}$ .*

In other words, the moduli space of boundary framed holomorphic structures on  $E$  can be identified with the moduli space of boundary framed Hermitian Yang-Mills connection on  $(E, H)$ .

In the special case when  $\bar{X}$  is the closure of a strictly pseudoconvex domain (with smooth boundary) in  $\mathbb{C}^n$ , Donaldson states the following result which gives an interesting geometric interpretation of the quotient  $\mathcal{C}^\infty(\partial\bar{X}, \text{GL}(r, \mathbb{C}))/\mathcal{O}^\infty(\bar{X}, \text{GL}(r, \mathbb{C}))$  of the group of smooth maps  $\partial\bar{X} \rightarrow \text{GL}(r, \mathbb{C})$  by the subgroup formed by those such maps which extend smoothly and formally holomorphically to  $\bar{X}$ :

**Corollary 3.** *Let  $\mathcal{O}^\infty(\bar{X}, \text{GL}(r, \mathbb{C}))$  be the group of smooth, formally holomorphic  $\text{GL}(r, \mathbb{C})$ -valued maps on  $\bar{X}$ , identified with a subgroup of  $\mathcal{C}^\infty(\partial\bar{X}, \text{GL}(r, \mathbb{C}))$  via the restriction map.*

*There exists a natural bijection between the moduli space of boundary framed Hermitian Yang-Mills connections on the trivial  $U(r)$ -bundle on  $\bar{X}$  and the quotient  $\mathcal{C}^\infty(\partial\bar{X}, \mathrm{GL}(r, \mathbb{C}))/\mathcal{O}^\infty(\bar{X}, \mathrm{GL}(r, \mathbb{C}))$ .*

The idea of proof: Taking into account Corollary 2, it suffices to construct a bijection between the quotient  $\mathcal{C}^\infty(\partial\bar{X}, \mathrm{GL}(r, \mathbb{C}))/\mathcal{O}^\infty(\bar{X}, \mathrm{GL}(r, \mathbb{C}))$  and the moduli space of boundary framed holomorphic structures on the trivial differentiable bundle  $\bar{X} \times \mathbb{C}^r$ . The construction is very natural: one maps the congruence class  $[f]$  of a smooth map  $f : \partial\bar{X} \rightarrow \mathrm{GL}(r, \mathbb{C})$  to the gauge class of the pair (the trivial holomorphic structure on  $\bar{X} \times \mathbb{C}^r, f$ ). The main difficulty is to prove the surjectivity of the map obtained in this way. This follows from the following existence result:

**Proposition 4.** *Let  $\bar{X}$  be the closure of a strictly pseudoconvex domain (with smooth boundary) in  $\mathbb{C}^n$  and  $\mathcal{E}$  be a smooth, topologically trivial holomorphic bundle on  $\bar{X}$ . Then  $\mathcal{E}$  admits a global smooth trivialization on  $\bar{X}$  which is holomorphic on  $X$ .*

The statement follows using Grauert's classification theorem for bundles on Stein manifolds and the following extension theorem, which is proved in [Do] only for  $n = 2$ :

**Proposition 5.** *Let  $\bar{X}$  be the closure of a relatively compact strictly pseudoconvex domain (with smooth boundary) in  $\mathbb{C}^n$  and  $\mathcal{E}$  be a smooth, topologically trivial holomorphic bundle on  $\bar{X}$ . Then  $\mathcal{E}$  extends holomorphically to an open neighborhood  $U$  of  $\bar{X}$  in  $\mathbb{C}^n$ .*

In my talk I will explain the idea of proof of the following general extension theorem (see [T]):

**Theorem 6.** *Let  $M$  be a complex manifold,  $X \subset M$  an open submanifold of  $M$  whose closure  $\bar{X}$  has smooth, strictly pseudoconvex boundary in  $M$ . Let  $G$  be a complex Lie group,  $\pi : Q \rightarrow M$  a differentiable principal  $G$ -bundle on  $M$  and  $J$  a holomorphic structure on the restriction  $\bar{P} := Q|_{\bar{X}}$ .*

*There exists an open neighborhood  $M'$  of  $\bar{X}$  in  $M$  and a holomorphic structure  $J'$  on  $Q|_{M'}$  which extends  $J$ .*

The proof uses methods and techniques introduced in [HiNa] and [Ca1].

In the special case when  $M = \mathbb{C}^n$  and  $G = \mathrm{GL}(r, \mathbb{C})$  one obtains as corollary Proposition 5 (and hence Corollary 3) in full generality. Moreover, one also obtains the following generalization of this corollary:

**Theorem 7.** *Let  $G = K^\mathbb{C}$  be the complexification of a compact Lie group  $K$ ,  $\bar{X}$  be a compact Stein manifold with boundary and  $g$  be a Hermitian metric  $g$  on  $\bar{X}$ . The moduli space of boundary framed Hermitian Yang-Mills connections on the trivial  $K$ -bundle on  $(\bar{X}, g)$  can be identified with the quotient  $\mathcal{C}^\infty(\partial\bar{X}, G)/\mathcal{O}^\infty(\bar{X}, G)$ .*

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