



International
Scientific Conference



Algebraic and Geometric Methods of Analysis



Devoted to 160 anniversary of
Dvytro Grave
(25.08.1863 - 19.12.1939)
Academician of the Ukrainian
Academy of Sciences, the
first director of the Institute of
Mathematics of NAS of Ukraine

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LIST OF TOPICS

- Algebraic methods in geometry
- Differential geometry in the large
- Geometry and topology of differentiable manifolds
- General and algebraic topology
- Dynamical systems and their applications
- Geometric and topological methods in natural sciences
- Geometric problems in mathematical analysis

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Twisted Sasaki metric on the unit tangent bundle and harmonicity

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Let (M^n, g) be an n -dimensional Riemannian manifold, TM^n be its tangent bundle, $\mathfrak{X}(M^n)$ be the Lie algebra of smooth vector fields of a Riemannian manifold (M^n, g) , ∇ be the Levi-Civita connection on M^n . The standard metric on the tangent bundle of Riemannian manifold (M^n, g) is the Sasaki metric [7]. It can be completely defined by scalar products of various combinations of vertical and horizontal lifts of vector fields. The Sasaki metric weakly inherits the base manifold properties. That is why the rigidity of Sasaki metric motivates many authors consider various deformations of Sasaki metric (see [1], [3] and others).

Belarbi L. and El Hendi H. introduce in [2] the twisted Sasaki metric on the tangent bundle TM as a new natural metric non-rigid on TM . The twisted Sasaki metric is defined as follows.

Definition 1. [2] Let (M^n, g) be a Riemannian manifold and $\delta, \varepsilon : M^n \rightarrow \mathbb{R}$ be strictly positive smooth functions. On the tangent bundle TM^n , we define a twisted Sasaki metric noted $G^{\delta, \varepsilon}$ by

$$\begin{aligned} G_{(x, \xi)}^{\delta, \varepsilon}(X^h, Y^h) &= \delta(x)g_x(X, Y), \\ G_{(x, \xi)}^{\delta, \varepsilon}(X^h, Y^v) &= 0, \\ G_{(x, \xi)}^{\delta, \varepsilon}(X^v, Y^v) &= \varepsilon(x)g_x(X, Y). \end{aligned}$$

for all vector fields $X, Y \in \mathfrak{X}(M^n)$ and $(x, \xi) \in TM^n$.

Note that, if $\delta = \varepsilon = 1$, then $G^{\delta, \varepsilon}$ is the Sasaki metric [7]. If $\delta = 1$, then $G^{\delta, \varepsilon}$ is the vertical rescaled metric (see [3], [4]).

For a unit vector field ξ on a compact Riemannian manifold (M, g) , Gerrit Weigmink [8] considered a very natural geometric functional

$$\int_M \|A_\xi\|^2 dVol(M),$$

where $\|A_\xi\|$ is a norm of the Nomizu operator $A_\xi X = -\nabla_X \xi$, i.e. $\|A_\xi\|^2 = \sum_{i=1}^n g(A_\xi e_i, A_\xi e_i)$ relative to some orthonormal frame (e_1, \dots, e_n) . It was proved, that this functional is unbounded from above. The critical points of this functional was called *harmonic unit vector fields*. G. Wigink proved, that a unit vector field ξ on compact Riemannian manifold is harmonic if and only if

$$\bar{\Delta}\xi = \|A_\xi\|^2 \xi,$$

where $\bar{\Delta}\xi$ is rough Laplacian (or Bochner Laplacian) of the field ξ defined as $\bar{\Delta}\xi = -\text{trace} \nabla^2 \xi$, where $\nabla_{X,Y}^2 = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$.

On the other hand (see [5]), the energy of a *mapping* $\phi : (M^n, g) \rightarrow (N^k, h)$ between Riemannian manifolds is defined as

$$E(\phi) := \frac{1}{2} \int_M |d\phi|^2 dVol_M.$$

The mapping ϕ is called *harmonic* if it is the critical point of the energy functional. It was proved that the mapping ϕ is harmonic if and only if the divergence of its differential vanishes, or equivalently its tension field $\tau(\phi) = \text{div}(d\phi)$ vanishes identically, where $|d\phi|$ is a norm of 1-form $d\phi$ in the cotangent bundle T^*M^n . Supposing on T_1M^n the Sasaki metric g_S , a unit vector field ξ as a mapping $\xi : (M^n, g) \rightarrow (T_1M^n, g_S)$ defines a *harmonic map* if and only if it is *harmonic* and, in addition, $\sum_{i=1}^n R(\xi, A_\xi e_i)e_i = 0$ relative to some orthonormal frame $\{e_i\}_{i=1}^n$.

In the present research we define the twisted Sasaki metric [2] on the unit tangent bundle T_1M^n of n -dimensional Riemannian manifold (M^n, g) , obtain Kowalski-type formulas, calculate the tension field of the mapping $\xi : M^n \rightarrow (T_1M^n, G^{\delta, \varepsilon})$. As a main result, for twisted Sasaki metric $G^{\delta, \varepsilon}$ on the unit tangent bundle T_1M^n of n -dimensional Riemannian manifold (M^n, g) we obtain the necessary and sufficient conditions for harmonicity of left-invariant unit vector field ξ and mapping $\xi : M^n \rightarrow (T_1M^n, G^{\delta, \varepsilon})$.

Theorem 2. *Unit vector field ξ is harmonic on n -dimensional Riemannian manifold (M^n, g) equipped with twisted Sasaki metric $G^{\delta, \varepsilon}$ on the unit tangent bundle T_1M^n if and only if*

$$\bar{\Delta}\xi + \frac{1}{\varepsilon}A_\xi(\nabla\varepsilon) = \|A_\xi\|^2\xi. \quad (1)$$

Harmonic unit vector field ξ defines a harmonic mapping $\xi : M^n \rightarrow (T_1M^n, G^{\delta, \varepsilon})$ on n -dimensional Riemannian manifold (M^n, g) equipped with twisted Sasaki metric $G^{\delta, \varepsilon}$ on the unit tangent bundle T_1M^n if and only if

$$2\varepsilon \cdot \text{trace}(Hm_\xi) + (n-2)\nabla\delta + \|A_\xi\|^2\nabla\varepsilon = 0. \quad (2)$$

As an examples, we consider the necessary and sufficient conditions for harmonicity of left-invariant unit vector field ξ and harmonic mapping $\xi : M^3 \rightarrow (T_1M^3, G^{\delta, \varepsilon})$ on 3-dimensional unimodular Lie group equipped with twisted Sasaki metric on the unit tangent bundle T_1M^3 , using orthonormal frame of Milnor J. [6]. In addition, we consider some examples of deformations that preserves existence harmonic left-invariant unit vector fields ξ and harmonic mapping $\xi : M^3 \rightarrow (T_1M^3, G^{\delta, \varepsilon})$ on 3-dimensional unimodular Lie groups with the left invariant metric.

REFERENCES

- [1] Abbassi MTK, Sarih M. On Riemannian g -natural metrics of the form $ag^s + bg^h + cg^v$ on the Tangent Bundle of a Riemannian Manifold (M, g) . *Mediterranean Journal of Mathematics* 2005; 2(1): 19-43.
- [2] Belarbi L, El Hendi H. Geometry of Twisted Sasaki Metric. *Journal of Geometry and Symmetry in Physics*, 53: 1-19, 2019.
- [3] Cheeger J, Gromoll D. On the structure of complete manifolds of nonnegative curvature. *Annals of Mathematics*, 96: 413-443, 1972.
- [4] Dida HM, Hathout F, Azzouz A. On the geometry of the tangent bundle with vertical rescaled metric. *Communications Faculty Of Science University of Ankara Series A1 Mathematics and Statistics*, 68(1): 222-235, 2019.
- [5] Eells J, Lemaire L. A Report on Harmonic Maps. *Bulletin of London Mathematical Society*, 10: 1-68, 1978.
- [6] Milnor J. Curvatures of left invariant metrics on Lie groups. *Advances in Mathematics*, 21: 293-329, 1976.
- [7] Sasaki S. On the differential geometry of tangent bundles of Riemannian manifolds II. *Tokyo Journal of Mathematics*, 14(2): 146-155, 1962.
- [8] Wjiegminck G. Total bending of vector fields on Riemannian manifolds. *Mathematische Annalen*, 303: 325-344, 1995.

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