

International scientific conference  
**«Algebraic and geometric  
methods of analysis»**

Book of abstracts



May 30 - June 4, 2018,  
Odesa,  
Ukraine

<https://www.imath.kiev.ua/~topology/conf/agma2018>

## A note on similarity of matrices

**Volodymyr M. Prokip**

(IAPMM NAS of Ukraine, L'viv, Ukraine)

*E-mail:* v.prokip@gmail.com

Let  $\mathbb{F}$  be a field of characteristic 0. Denote by  $\mathbb{F}_{m \times n}$  the set of  $m \times n$  matrices over  $\mathbb{F}$  and by  $\mathbb{F}_{m \times n}[x_1, x_2, \dots, x_n]$  the set of  $m \times n$  matrices over the polynomial ring  $\mathbb{F}[x_1, x_2, \dots, x_n]$ . Denote by  $GL(n, \mathbb{F})$  the group of invertible matrices in  $\mathbb{F}_{n \times n}$ . In what follows, we denote by  $I_n$  the  $n \times n$  identity matrix and by  $0_{n,k}$  the zero  $m \times n$  matrix. The Kronecker product of matrices  $A = [a_{ij}] \in \mathbb{F}_{m \times n}$  and  $B$  is denoted by

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}.$$

Matrices  $A \in \mathbb{F}_{n \times n}$  and  $B \in \mathbb{F}_{n \times n}$  are said to be similar, if there is a nonsingular matrix  $P \in \mathbb{F}_{n \times n}$  such that  $A = PBP^{-1}$ . Two tuples of  $n \times n$  matrices over  $\mathbb{F}$

$$\mathbf{A} = \{A_1, A_2, \dots, A_k\} \quad \text{and} \quad \mathbf{B} = \{B_1, B_2, \dots, B_k\}$$

are said to be simultaneously similar if there exists a matrix  $U \in GL(n, \mathbb{F})$  such that

$$A_i = U^{-1}B_iU \quad \text{for all} \quad i = 1, 2, \dots, k.$$

The task of classifying square matrices up to similarity is one of the core and oldest problems in linear algebra (see [2], [4], [5], [9], [7] and references therein), and it is generally acknowledged that it is also one of the most hopeless problems already for  $k = 2$ . Standard approaches for deciding similarity depend upon the Jordan canonical form, the invariant factor algorithm and the Smith form, or the closely related rational canonical form. In numerical linear algebra, this leads to deep algorithmic problems, unsolved even up to this date, that are caused by numerical instabilities in solving eigenvalue problems or by the inability to effectively compute sizes of the Jordan blocks or degrees of invariant factors, if the matrix entries are not known precisely.

The purpose of this report is to give an exposition of geometric ideas to solution of similarity of matrices over a field. We propose new necessary and sufficient conditions under which matrices  $A \in \mathbb{F}_{n \times n}$  and  $B \in \mathbb{F}_{n \times n}$  are similar.

Chris Byrnes and Michael Gauger [6] derived a new type of rank conditions for algebraically deciding similarity of arbitrary pairs of matrices over a field  $\mathbb{F}$ . Their main result is that two matrices  $A, B \in \mathbb{F}_{n \times n}$  are similar if and only if the following two conditions hold:

1. The characteristic polynomials coincide, i.e.  $\det(I_n x - A) = \det(I_n x - B)$ .
2.  $\text{rank}(I_n \otimes A - A \otimes I_n) = \text{rank}(I_n B - B \otimes I_n) = \text{rank}(B \otimes I_n - I_n \otimes A)$ .

In subsequent work by J.D. Dixon [1] it was shown that the first condition on the characteristic polynomials is superfluous, so that similarity can be solely decided based on rank computations. He proved that two matrices  $A, B \in \mathbb{F}_{n \times n}$  are similar if and only if

$$r^2(A, B) = r(A, A)r(B, B).$$

Here  $r(A, B) = \text{rank}(B \otimes I_n - I_n \otimes A)$  and similarly for  $r(A, A)$  and  $r(B, B)$ .

Over an algebraically closed field, S. Friedland [3] showed the closely related linear dimension inequality

$$2 \dim \text{Ker}(B \otimes I_n - I_n \otimes A) \leq \dim \text{Ker}(A \otimes I_n - I_n \otimes A) + \dim \text{Ker}(B \otimes I_n - I_n \otimes B)$$

and proved that equality holds if and only if matrices  $A$  and  $B$  are similar.

For given two matrices  $A, B \in \mathbb{F}_{n \times n}$  we define the matrix

$$M = [ A \otimes I_n - I_n \otimes B ] \in \mathbb{F}_{n^2 \times n^2}.$$

For matrix  $M$  there exist a matrix  $W \in GL(n^2, \mathbb{F})$  such that

$$MW = H_M = \begin{bmatrix} 0_{l,1} & 0_{l,n^2-1} \\ H_1 & 0_{m_1,n^2-1} \\ H_2 & 0_{m_2,n^2-2} \\ \dots & \dots \\ H_k & 0_{m_k,r} \end{bmatrix}$$

is the lower block-triangular matrix and  $l$  is the number of first zero rows of matrix  $M$ . The matrices  $H_i$  are defined as follows:

$$H_1 = \begin{bmatrix} 1 \\ * \end{bmatrix} \in M_{m_1,1}(\mathbb{F}), \quad H_2 = \begin{bmatrix} 0 & 1 \\ * & * \end{bmatrix} \in M_{m_2,2}(\mathbb{F}), \quad H_r = \begin{bmatrix} 0 & \dots & 0 & 1 \\ * & * & * & * \end{bmatrix} \in M_{m_r,n^2-r}(\mathbb{F})$$

and  $l + m_1 + m_2 + \dots + m_k = n^2$ .

The lower block-triangular matrix  $H_M$  is called the Hermite normal form of the matrix  $M$  and the form  $H_M$  is uniquely determined by  $M$  (see [8]). It is evident that  $\text{rank } H_M = n^2 - r$ .

Let  $x_1, x_2, \dots, x_r$  be independent variables. Consider the vector

$$W \begin{bmatrix} 0_{n^2-r,1} \\ x_1 \\ \vdots \\ x_r \end{bmatrix} = \begin{bmatrix} W_1(x_1, \dots, x_r) \\ W_2(x_1, \dots, x_r) \\ \vdots \\ W_n(x_1, \dots, x_r) \end{bmatrix} = \begin{bmatrix} W_1(\bar{x}) \\ W_2(\bar{x}) \\ \vdots \\ W_n(\bar{x}) \end{bmatrix} \in \mathbb{F}_{n^2,1}[x_1, x_2, \dots, x_r],$$

where  $W_i(\bar{x}) = W_i(x_1, \dots, x_r) \in \mathbb{F}_{n,1}[x_1, x_2, \dots, x_r]$ . Put

$$W(\bar{x}) = [W_1(\bar{x}) \quad W_2(\bar{x}) \quad \dots \quad W_n(\bar{x})] \in \mathbb{F}_{n,n}[x_1, x_2, \dots, x_r].$$

**Theorem 1.** *If matrices  $A, B \in \mathbb{F}_{n \times n}$  are similar over  $\mathbb{F}$  then  $\text{rank } M \leq n^2 - n$ .*

**Theorem 2.** *Matrices  $A, B \in \mathbb{F}_{n \times n}$  are similar over  $\mathbb{F}$  if and only if matrix  $W(\bar{x})$  is nonsingular.*

**Corollary 3.** *If  $A = B \in \mathbb{F}_{n \times n}$  then matrix  $W(\bar{x})$  is nonsingular.*

#### REFERENCES

- [1] J.D. Dixon. An isomorphism criterion for modules over a principal ideal domain. *Linear and Multilinear Algebra*, 8: 69-72, 1979.
- [2] Yu.A. Drozd. Representations of commutative algebras. *Functional Analysis and Its Applications*, 6(4): 286-288, 1972.
- [3] S. Friedland. Analytic similarity of matrices. *In Algebraic and Geometric Methods in Linear Systems Theory*, (eds. C.I. Byrnes and C.F. Martin), Lectures in Applied Math. Amer. Math. Soc., 18: 43-85, 1980.
- [4] S. Friedland. Simultaneous similarity of matrices. *Adv. Math.*, 50: 189-265, 1983.
- [5] S. Friedland. *Matrices: Algebra, Analysis and Applications*. Singapore: World Scientific Publishing Co., 2015.
- [6] M.A. Gauger and C.I. Byrnes. Characteristic free, improved decidability criteria for the similarity problem. *Linear and Multilinear Algebra*, 5: 153-158, 1977.
- [7] K.D. Ikramov. How to check whether given square matrices are congruent? *Zapiski Nauchnykh Seminarov POMI*. 439: 99-106, 2015.
- [8] E.D. Nering. *Linear algebra and matrix theory*. New York & London: John Wiley and Sons, Inc., 1963.
- [9] V.V. Sergeichuk. Canonical matrices for linear matrix problems. *Linear Algebra Appl.*, 317: 53-102, 2000.

<b>V. Lomadze</b> <i>A purely algebraic construction of Schwartz distributions</i>	<b>34</b>
<b>S. Maksymenko</b> <i>Deformation of functions on orientable surfaces by symplectic diffeomorphisms</i>	<b>36</b>
<b>Martseniuk V., Bondarenko A., Gefter S.</b> <i>Application the <math>p</math>-adic topology on <math>Z</math> for study determinants of infinite order with integer coefficients</i>	<b>37</b>
<b>K. Matsumoto</b> <i>Warped product semi-slant submanifolds in locally conformal Kaehler manifolds, II</i>	<b>38</b>
<b>Mormul P.</b> <i>What points in Goursat Monster Tower are strongly nilpotent</i>	<b>39</b>
<b>Mukhamadiev F. G.</b> <i>The local density and the local weak density of <math>N_7^\varphi</math>-kernel of a topological spaces</i>	<b>40</b>
<b>Tetiana V. Obikhod</b> <i>Topological invariants and microscopic quantum description</i>	<b>41</b>
<b>A. Ozkan</b> <i>On the misconceptions of 10th grade students about analytical geometry</i>	<b>42</b>
<b>Parasyuk I. O.</b> <i>Hyperbolic quasiperiodic motion of charged particle on 2-sphere</i>	<b>43</b>
<b>E. Petrov</b> <i>Representing trees of finite ultrametric spaces and weak similarities</i>	<b>45</b>
<b>Prishlyak A., Hladysh B.</b> <i>Functions with three critical points on closed non-oriented 3-manifolds</i>	<b>47</b>
<b>V.M. Prokip</b> <i>A note on similarity of matrices</i>	<b>49</b>
<b>Romakina L.N.</b> <i>Inversion with respect to an elliptic cycle of a hyperbolic plane of positive curvature</i>	<b>51</b>
<b>Romaniv A. M.</b> <i>On the relationship between the Smith normal forms of matrices and of their least common multiple</i>	<b>53</b>
<b>Ryazanova O.</b> <i>Estimates for the surface with given average curvature</i>	<b>54</b>
<b>Samokhvalov S. E.</b> <i>Kinematic renormalization of energy in the gravity</i>	<b>55</b>
<b>Savchenko A.</b> <i>Fuzzy metrization of the spaces of idempotent measures</i>	<b>56</b>
<b>Sazonova O.</b> <i>General form of the Maxwellian distribution with arbitrary density</i>	<b>57</b>
<b>Skuratovskii R.</b> <i>The commutator and centralizer of Sylow subgroups of alternating and symmetric groups, its minimal generating set</i>	<b>58</b>
<b>Sokhatsky F., Krainichuk H.</b> <i>Semi-lattice of varieties of quasigroups with linearity</i>	<b>59</b>
<b>Salimov R., Stefanchuk M.</b> <i>Asymptotic behavior of solutions to a nonlinear Beltrami equation</i>	<b>61</b>
<b>Mokritskaya T. P., Tushev A. V.</b> <i>On fractal characteristics of the micro-aggregate structure of loess soils</i>	<b>62</b>
<b>Vasilchenko A. N.</b> <i>Spaces of primitive elements in dual modules over Steenrod algebra</i>	<b>64</b>