

International  
Online Conference



**Algebraic  
and Geometric  
Methods of Analysis**

dedicate to the memory  
of Yuriy Trokhymchuk  
(17.03.1928-18.12.2019)

May 25-28, 2021  
Odesa, Ukraine

## LIST OF TOPICS

- Topological methods in analysis
- Geometric problems of complex and mathematical analysis
- Algebraic methods in geometry
- Differential geometry in the whole
- Geometry and topology of differentiable manifolds
- General and algebraic topology
- Geometric and topological methods in natural sciences

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# Elements of probability theory and measures with values in hypercomplex algebras

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In recent years, the expansion of probability theory and measure theory from real values to values in hypercomplex numbers are actively studied because of their possible applications in mathematics and physics [1] – [5]. In this paper, we extend the notion of probability measure to the case where the measure takes values in the algebra of bihyperbolic numbers [6]. In addition, the concept of the real-valued measure is generalized to the quaternionic-valued measure [7].

The bihyperbolic numbers forms a 4-dimensional algebra over the field of real numbers  $\mathbb{W}_4 = \{a_0 + a_1e + a_2f + a_3g, a_i \in \mathbb{R}, i = 0, 1, 2, 3\}$  with basis  $\{1, e, f, g\}$  and the following multiplications  $e^2 = f^2 = g^2 = 1, ef = fe = g, eg = ge = f, fg = gf = e$ .

**Lemma 1.** [8] *Any bihyperbolic number  $\alpha$  can be represented as  $\alpha = r_1i_1 + r_2i_2 + r_3i_3 + r_4i_4$ , where  $i_k$  are idempotents of algebra  $\mathbb{W}_4$ ,  $r_k \in \mathbb{R}, k = 1, 2, 3, 4$ .*

We define on  $\mathbb{W}_4$  the relation of partial order  $\preceq$  such as  $\alpha \preceq \beta \iff \beta - \alpha \in \mathbb{W}_4^+ = \{x_1i_1 + x_2i_2 + x_3i_3 + x_4i_4 \mid x_k \geq 0, k = 1, 2, 3, 4\}$ . If  $\alpha \preceq \beta$  but  $\alpha \neq \beta$ , we denote  $\alpha \prec \beta$ . Let us denote by  $A_x$ , the set of all bihyperbolic numbers which are not  $\mathbb{W}_4$ -comparable with  $x \in \mathbb{W}_4$ .

**Definition 2.** The  $\mathbb{W}_4$ -valued modulus of a bihyperbolic number  $\alpha = r_1i_1 + r_2i_2 + r_3i_3 + r_4i_4$  is said to be  $|\alpha|_{\mathbb{W}_4} = |r_1i_1 + r_2i_2 + r_3i_3 + r_4i_4|_{\mathbb{W}_4} = |r_1|i_1 + |r_2|i_2 + |r_3|i_3 + |r_4|i_4 \in \mathbb{W}_4^+$ , where  $|r_1|, |r_2|, |r_3|, |r_4|$  are ordinary modules of real numbers.

**Definition 3.** Let  $(\Omega, \Sigma)$  be a measurable space. The function  $P_{\mathbb{W}_4} : \Sigma \rightarrow \mathbb{W}_4$  is called a  $\mathbb{W}_4$ -valued probability (or bihyperbolic probability) on the  $\sigma$ -algebra of events  $\Sigma$ , if the following conditions hold: 1)  $P_{\mathbb{W}_4}(A) \succcurlyeq 0, \forall A \in \Sigma$ ; 2)  $P_{\mathbb{W}_4}(\Omega) = \zeta$ , where  $\zeta = 1, i_1, i_2, i_3, i_4$ ; 3) For any sequence  $\{A_n, n \geq 1\} \subset \Sigma$  of pairwise incompatible random events we have  $P_{\mathbb{W}_4}(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P_{\mathbb{W}_4}(A_n)$ .

The triplet  $(\Omega, \Sigma, P_{\mathbb{W}_4})$  is called a  $\mathbb{W}_4$ -probability space.

Each  $\mathbb{W}_4$ -valued probability measure can be written in the form  $P_{\mathbb{W}_4}(A) = P_1(A)i_1 + P_2(A)i_2 + P_3(A)i_3 + P_4(A)i_4$ , where  $P_1(A), P_2(A), P_3(A), P_4(A)$  are probabilities.

The topology induced by the bihyperbolic norm generates the Borel  $\sigma$ -algebra  $\mathfrak{B}_{\mathbb{W}_4}$  in  $\mathbb{W}_4$ .

**Definition 4.** Let  $(\Omega, \Sigma, P_{\mathbb{W}_4})$  be a  $\mathbb{W}_4$ -probability space. A function  $X_{\mathbb{W}_4}(\omega) : \Omega \rightarrow \mathbb{W}_4$  such as  $X_{\mathbb{W}_4}^{-1}(A) \in \Sigma$  for each open set  $A$  in  $\mathbb{W}_4$  is called a  $\mathbb{W}_4$ -valued random variable.

Each  $\mathbb{W}_4$ -valued random variable  $X_{\mathbb{W}_4}(\omega)$  can be written in the following form  $X_{\mathbb{W}_4}(\omega) = X_1(\omega)i_1 + X_2(\omega)i_2 + X_3(\omega)i_3 + X_4(\omega)i_4$ , where  $X_1(\omega), X_2(\omega), X_3(\omega), X_4(\omega)$  are  $\mathbb{R}$ -random variables on  $\Omega$ .

**Theorem 5.** *The  $\mathbb{W}_4$ -valued function  $X_{\mathbb{W}_4}(\omega)$  on a measurable space  $(\Omega, \Sigma)$  is a  $\mathbb{W}_4$ -valued random variable if and only if  $\{\omega \in \Omega \mid X_{\mathbb{W}_4}(\omega) \prec x \text{ or } X_{\mathbb{W}_4}(\omega) \in A_x\} \in \Sigma$  for all  $x \in \mathbb{W}_4$ .*

**Theorem 6.** Let  $X_{\mathbb{W}_4}(\omega)$  be a  $\mathbb{W}_4$ -valued random variable on  $(\Omega, \Sigma, P_{\mathbb{W}_4})$ . For  $\forall x \in \mathbb{W}_4$  the following conditions are equivalent:  $\{\omega \in \Omega | X_{\mathbb{W}_4}(\omega) \preceq x\} \in \Sigma$ ;  $\{\omega \in \Omega | X_{\mathbb{W}_4}(\omega) \succ x \text{ or } X_{\mathbb{W}_4}(\omega) \in A_x\} \in \Sigma$ ;  $\{\omega \in \Omega | X_{\mathbb{W}_4}(\omega) \succcurlyeq x \text{ or } X_{\mathbb{W}_4}(\omega) \in A_x\} \in \Sigma$ ;  $\{\omega \in \Omega | X_{\mathbb{W}_4}(\omega) \prec x\} \in \Sigma$ .

The algebra of quaternions is a structure of the form  $\mathbb{H} = \{a_0 + a_1i + a_2j + a_3k, a_i \in \mathbb{R}, i = 0, 1, 2, 3\}$ , where  $i^2 = j^2 = k^2 = -1$ ,  $ij = -ji = k$ ,  $jk = -kj = i$ ,  $ki = -ik = j$ .

**Definition 7.** Let  $\mathfrak{M}$  be a  $\sigma$ -algebra of subsets of a non-empty set  $X$ . A quaternionic measure  $\omega$  on a measurable space  $(X, \mathfrak{M})$  is a quaternion-valued function on  $\mathfrak{M}$  such that for any collection of sets  $\{A_n, n \in \mathbb{N}\} \subset \mathfrak{M}$  that  $A_n \cap A_m = \emptyset$  whenever  $n \neq m$  we have  $\omega(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \omega(A_n)$ .

**Definition 8.** The function of the sets  $\text{var}[\omega](A) := \sup \sum_{n=1}^{\infty} |\omega(A_n)|$  is defined on the  $\mathfrak{M}$ , where the supremum is taken for all partitions of  $A$ , we call the complete variation  $\omega$ .

It is clear that  $|\omega(A)| \leq \text{var}[\omega](A)$ .

**Theorem 9.** The total variation  $\text{var}[\omega]$  of a quaternionic measure  $\omega$  on a measurable space  $(X, \mathfrak{M})$  is a positive measure on  $(X, \mathfrak{M})$ .

**Theorem 10.** If  $\omega$  is a quaternionic measure on a measurable space  $(X, \mathfrak{M})$ , then  $\text{var}[\omega](X) < \infty$ .

**Definition 11.** Let  $\mu$  be a positive measure and  $\omega$  be a quaternionic measure on a measurable space  $(X, \mathfrak{M})$ . We say that  $\omega$  is absolutely continuous with respect to  $\mu$  if  $\mu(A) = 0$  implies  $\omega(A) = 0$  for  $A \in \mathfrak{M}$ . We write  $\omega \ll \mu$ .

**Definition 12.** Given a quaternionic measure  $\omega$  on a measurable space  $(X, \mathfrak{M})$ , assume that there is a set  $F \in \mathfrak{M}$  such that  $\omega(A) = \omega(A \cap F)$  for every  $A \in \mathfrak{M}$ , we say that  $\omega$  is concentrated on  $F$ . This is equivalent to say that  $\omega(A) = 0$  whenever  $A \cap F = \emptyset$ .

Let  $\omega_1, \omega_2$  be quaternionic measures on  $\mathfrak{M}$  and suppose there exist a pair of disjoint sets  $F, G$  such that  $\omega_1$  is concentrated on  $F$  and  $\omega_2$  is concentrated on  $G$ . Then we say that  $\omega_1$  and  $\omega_2$  are mutually singular, and write  $\omega_1 \perp \omega_2$ .

**Theorem 13.** Let  $\lambda$  be a signed real  $\sigma$ -finite measure on a measurable space  $(X, \mathfrak{M})$  and let  $\omega$  be a quaternionic measure on  $(X, \mathfrak{M})$ . Then there exists a unique pair of quaternionic measures  $\omega_a$  and  $\omega_s$  such that  $\omega = \omega_a + \omega_s$ ,  $\omega_a \ll \lambda$ ,  $\omega_s \perp \lambda$ . The pair  $\omega_a, \omega_s$  is called the Lebesgue decomposition of  $\omega$  w.r.t.  $\lambda$ , where  $\omega_a$  is the absolutely continuous part and  $\omega_s$  is the singular part of the decomposition.

**Theorem 14.** (Radon-Nikodym). Let  $\mu$  be a positive  $\sigma$ -finite measure on a measurable space  $(X, \mathfrak{M})$ , let  $\omega$  be a quaternionic measure on  $(X, \mathfrak{M})$  and let  $\omega_a$  be absolutely continuous part of the Lebesgue decomposition of  $\omega$  w.r.t.  $\mu$ . Then there is a measurable quaternionic function  $h(x)$  on  $X$  such that for every set  $A \in \mathfrak{M}$   $\omega_a(A) = \int_A h(x) d\mu$ , where  $h(x)$  is uniquely defined up to a  $\mu$ -null set.

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