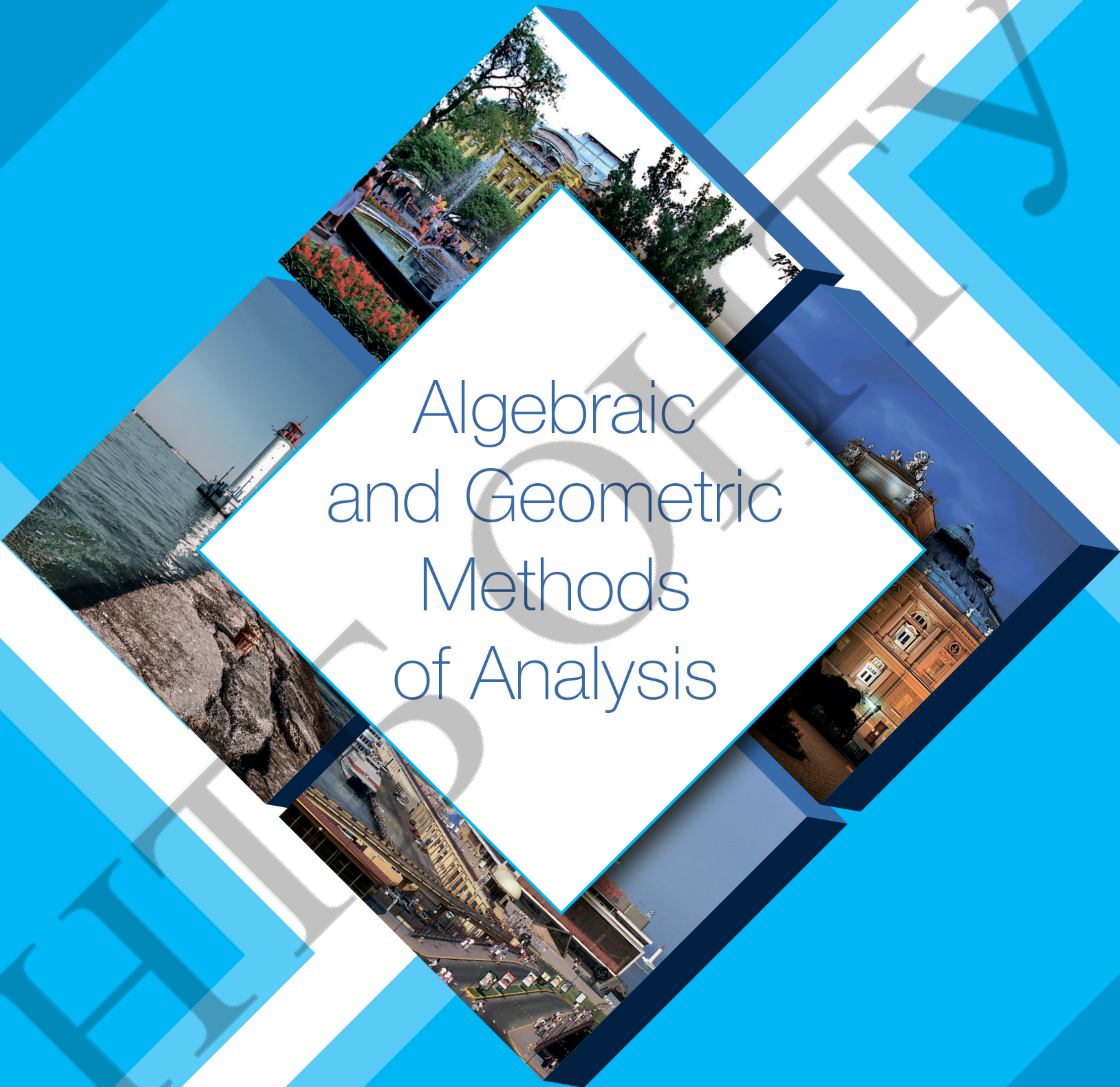
International
Scientific Conference

Algebraic and Geometric Methods of Analysis



27-30 May 2024

Odesa, Ukraine

The purpose of this conference is to bring together researchers in geometry, topology, algebra, analysis and dynamical systems and to provide for them a forum to present their recent work to colleagues from different nationalities. This way we aim to stimulate discussion about the latest findings in geometrical and topological methods in analysis and to increase international collaboration.

The conference continues the traditional annual conference «Geometry in Odesa» holding from 2004, and hosted by Odesa National University of Technology (Odesa National Academy of Food Technologies till 2021). From 2017 the conference was renamed to «Algebraic and geometric methods of analysis» (AGMA).

The Conference languages: Ukrainian and English.

List of topics

- Algebraic methods in geometry
- Differential geometry in the large
- Geometry and topology of differentiable manifolds
- General and algebraic topology
- Dynamical systems and their applications
- Geometric and topological methods in natural sciences
- Geometric problems in mathematical analysis

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(c) **Ultrametric inequality:** For all $x, y, z \in X$:

$$\overline{d}(x,z) = \sup\{d_A(x,z) \mid A \subseteq X\}$$

and

$$\overline{d}(x,z) \leq \sup\{\max\{d_A(x,y), d_A(y,z)\} \mid A \subseteq X\} \leq \max\{\overline{d}(x,y), \overline{d}(y,z)\}$$

since d_A satisfies the ultrametric inequality for all $A \subseteq X$.

(2) **Comparison $\overline{d} \leq d$:** For each $A \subseteq X$, we have $d_A \leq d$, thus:

$$\overline{d}(x,y) = \sup\{d_A(x,y) \mid A \subseteq X\} \leq d(x,y).$$

(3) **Greatest ultrapseudometric not exceeding d:** Suppose there exists another ultrapseudometric d' on X such that $d' \leq d$ and $d' \geq \overline{d}$. Then, for any $A \subseteq X$, $d_A \leq d'$, hence:

$$\overline{d} = \sup\{d_A \mid A \subseteq X\} \leq d'.$$

Therefore, $\overline{d}(x,y) = \sup\{d_A(x,y) \mid A \subseteq X\}$ is the greatest ultrapseudometric on X not exceeding d. □

We will discuss efficient algorithms for calculation of $\bar{d}$ for a given d on a finite set X.

References

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Action of derivations on polynomials and on Jacobian derivations

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Let $\mathbb{K}$ be an arbitrary field of characteristic zero. Denote by $A := \mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring, and by $R := \mathbb{K}(x_1, \ldots, x_n)$ the field of rational functions in n variables, respectively. A $\mathbb{K}$-linear map $D : A \to A$ is called a $\mathbb{K}$-derivation on A if $D(fg) = D(f)g + fD(g)$ for any $f, g \in A$. The vector space $W_n(\mathbb{K})$ (over $\mathbb{K}$) of all $\mathbb{K}$-derivation is a Lie algebra with respect to the Lie bracket $[D_1, D_2] = D_1D_2 - D_2D_1$, $D_1, D_2 \in W_n(\mathbb{K})$. Recall that every element $D \in W_n(\mathbb{K})$ can be uniquely written in the form

$$D = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}, f_i \in A.$$

The latter means that $W_n(\mathbb{K})$ is a free module of rank n over A with the free generators $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ (see, for example [3], [4]).

Every element D from $W_n(\mathbb{K})$ acts naturally on polynomials from A and on $W_n(\mathbb{K})$ itself (by multiplication). Recall that a polynomial $f \in A$ is a Darboux polynomial for a derivation $D \in W_n(\mathbb{K})$ if $D(f) = \lambda f$ for some $\lambda \in A$, the polynomial λ is called a cofactor for D. One can consider the Darboux polynomials as "eigenvectors" for the derivation D with polynomial "eigenvalues". These (non-constant) polynomials (if they do exist) play significant role in theory of differential equations because for a derivation $D = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}$ one can consider an autonomous system of differential equations of the form

$$\frac{dx_1}{dt} = f_1(x_1, \ldots, x_n), \ldots, \frac{dx_n}{dt} = f_n(x_1, \ldots, x_n)$$

and Darboux polynomials for D are very useful for searching solutions of this system (see, for example, [1], [2]).

We study normalizers of polynomials and derivations under the action of $W_n(\mathbb{K})$ on A and on itself (by multiplication) respectively. For any $f \in A$ one can consider the "normalizer" $N(f)$ in $W_n(\mathbb{K})$ of the form

$$N(f) = \{T \in W_n(\mathbb{K}) \mid T(f) = \lambda f \text{ for some } \lambda \in A\},$$

i.e. $N(f)$ is the set of all the derivations for which f is a Darboux polynomial. This normalizer is a subalgebra of the Lie algebra $W_n(\mathbb{K})$ and it acts on the principal ideal $(f) = Af$ of the ring A. The restriction $\widehat{N}(f)$ of the Lie algebra $N(f)$ on Af is characterized in the next statement.

Theorem 1. *The Lie algebra $\widehat{N}(f)$ is isomorphic to a subalgebra of the semidirect sum $W_n(\mathbb{K}) \leftthreetimes A$.*

Analogously for any $D \in W_n(\mathbb{K})$, one can consider the normalizer of D in $W_n(\mathbb{K})$ of the form

$$N(D) = \{T \in W_n(\mathbb{K}) \mid [T, D] = \lambda D \text{ for some } \lambda \in A\}$$

($N(D)$ is obviously the usual normalizer of the subalgebra AD in the Lie algebra $W_n(\mathbb{K})$). An analogous characterization of $N(D)$ is obtained.

Further, we consider more detailed the Lie algebra $W_2(\mathbb{K})$ and denote for convenience $A = \mathbb{K}[x, y]$. Let $f \in A$, $f \neq 0$. The polynomial f defines a derivation $D_f \in W_2(\mathbb{K})$ by the rule: $D_f(h) = \det J(f, h)$ for any $h \in \mathbb{K}[x, y]$ (here $J(f, h)$ is the Jacobi matrix for f and h). The derivation D_f is called the Jacobian derivation associated with the polynomial f. Note that all the Jacobian derivations form a subalgebra of $W_2(\mathbb{K})$ which coincides with the subalgebra $\mathfrak{sa}_2(\mathbb{K})$ consisting of all divergence-free derivations (see, for example [5]). If for some derivation $T \in W_2(\mathbb{K})$ there exists a Jordan chain consisting of polynomials

$$T(f_1) = \lambda f_1 + f_2, \ldots, T(f_{k-1}) = \lambda f_{k-1} + f_k, T(f_k) = \lambda f_k$$

for some $\lambda \in \mathbb{K}$, $k \geq 1$ then we prove the next statement

Theorem 2. *Let $T \in W_2(\mathbb{K})$ acts on polynomials $f_1, \ldots f_k$ by the rule*

$$T(f_1) = \lambda f_1 + f_2, \ldots, T(f_{k-1}) = \lambda f_{k-1} + f_k, T(f_k) = \lambda f_k$$

for some $\lambda \in \mathbb{K}$, $k \geq 1$. Then the equalities hold:

$$[T, D_{f_1}] = (\lambda - divT)D_{f_1} + D_{f_2},\ [T, D_{f_2}] = (\lambda - divT)D_{f_2} + D_{f_3}, \ldots,$$

$$[T, D_{f_k}] = (\lambda - divT)D_{f_k}.$$

The proof of this result is based on the next statement which is of independent interest.

Proposition 3. *Let $T \in W_2(\mathbb{K})$, $f \in \mathbb{K}[x, y]$ and $T(f) = g$ for some polynomial $g \in \mathbb{K}[x, y]$. Then $[T, D_f] = (-divT)D_f + D_g$. And conversely, if $[T, D_f] = (-divT)D_f + D_g$ for some $g \in A$, then $T(f) = g + c$ for some $c \in \mathbb{K}$.*

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Periodic point theorem for mappings contracting total pairwise distance

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We consider a new type of mappings in metric spaces so-called mappings contracting total pairwise distance on n points, see [1]. It is shown that such mappings are continuous. A theorem on the existence of periodic points for such mappings is proved and the classical Banach fixed-point theorem is obtained like a simple corollary as well as the fixed point theorem for mappings contracting perimeters of triangles.

Everywhere below by $|X|$ we denote the cardinality of the set X. Let (X, d) be a metric space, $|X| \geqslant 2$, and let x_1, x_2, ..., $x_n \in X$, $n \geqslant 2$. Denote by

$$S(x_1, x_2, \ldots, x_n) = \sum_{1 \leqslant i < j \leqslant n} d(x_i, x_j)$$

the sum of all pairwise distances between the points from the set $\{x_1, x_2, \ldots, x_n\}$, which we call *total pairwise distance*.

Definition 1. Let $n \geqslant 2$ and let (X, d) be a metric space with $|X| \geqslant n$. We shall say that $T\colon X \to X$ is a *mapping contracting total pairwise distance on n points* if there exists $\alpha \in [0, 1)$ such that the inequality

$$S(Tx_1, Tx_2, \ldots, Tx_n) \leqslant \alpha S(x_1, x_2, \ldots, x_n) \tag{1}$$

holds for all n pairwise distinct points $x_1, x_2, \ldots, x_n \in X$.

Note that the requirement for $x_1, x_2, \ldots, x_n \in X$ to be pairwise distinct is essential, which is confirmed by the following proposition.

Proposition 2. *Suppose that in Definition 1 inequality (1) holds for any n points $x_1, x_2, \ldots, x_n \in X$ with $|\{x_1, x_2, \ldots, x_n\}| = k$, where $2 \leqslant k \leqslant n-1$. Then T is a mapping contracting total pairwise distance on k points.*

Proposition 3. *Mapping contracting total pairwise distance on m points, $m \geqslant 2$, is a mapping contracting total pairwise distance on n points for all $n > m$.*

Proposition 4. *Mappings contracting total pairwise distance on n points are continuous.*