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Online Conference



**Algebraic
and Geometric
Methods of Analysis**

dedicate to the memory
of Yuriy Trokhymchuk
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LIST OF TOPICS

- Topological methods in analysis
- Geometric problems of complex and mathematical analysis
- Algebraic methods in geometry
- Differential geometry in the whole
- Geometry and topology of differentiable manifolds
- General and algebraic topology
- Geometric and topological methods in natural sciences

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Topology of spaces of smooth functions and gradient-like flows with prescribed singularities on surfaces

Elena Kudryavtseva

(Moscow State University, Faculty of Mechanics and Mathematics, Moscow Center of Fundamental and Applied Mathematics, Moscow, Russia)

E-mail: eakudr@mech.math.msu.su

Let M be a smooth orientable connected closed two-dimensional surface, and $f_0 \in \mathcal{F}(M)$ a function all whose critical points have types A_i, D_j, E_k . Consider the set $\mathcal{F} = \mathcal{F}(f_0)$ of all functions $f \in C^\infty(M)$ having the same types of local singularities as f_0 . Denote by $\mathcal{D}^0(M)$ the component of unity in the group $\mathcal{D}(M) = \text{Diff}^+(M)$ of orientation-preserving diffeomorphisms. The group $\mathcal{D}(\mathbb{R}) \times \mathcal{D}(M)$ acts on the space \mathcal{F} by “left-right changes of coordinates”.

We want to describe the topology of the space \mathcal{F} , equipped with the C^∞ -topology, and its decomposition into $\mathcal{D}^0(\mathbb{R}) \times \mathcal{D}^0(M)$ - and $\mathcal{D}^0(M)$ -orbits. This problem was solved by the author in the cases when either f_0 is a Morse function and $\chi(M) < 0$ [2, 3], or all critical points of f_0 have A_μ types, $\mu \in \mathbb{N}$ [4]. Topology of the $\mathcal{D}^0(M)$ -orbits was studied by S.I. Maksymenko [5] (allowing some other types of degenerate singularities) and by the author [2, 3, 4] (for A_μ -singularities).

For any function $f \in \mathcal{F}$, consider the set $\mathcal{C}_f := \{P \in M \mid df(P) = 0\}$ of its critical points. These critical points form five classes of topological equivalence (some classes may be empty):

$$\mathcal{C}_f^{\min} = \bigcup_{i \geq 1} A_{2i-1}^{+,+}(f), \quad \mathcal{C}_f^{\max} = \bigcup_{i \geq 1} A_{2i-1}^{+,-}(f), \quad \mathcal{C}_f^{\text{saddle}} = A_1^-(f) \cup \bigcup_{\eta=\pm} \bigcup_{i \geq 2} (A_{2i-1}^{-,\eta}(f) \cup D_{2i+1}^\eta(f)) \cup E_7^\eta(f),$$

$$\mathcal{C}_f^{\text{triv}} = \left(\bigcup_{i \geq 1, \eta=\pm} A_{2i}^\eta(f) \right) \cup \left(\bigcup_{j \geq 2} D_{2j}^+(f) \right) \cup E_6^+(f) \cup E_6^-(f) \cup E_8^+(f) \cup E_8^-(f), \quad \mathcal{C}_f^{\text{mult}} = \bigcup_{j \geq 2} D_{2j}^-(f),$$

i.e. the critical points of local minima, local maxima, saddle points, quasi- and multy-saddle points, respectively. Here $A_i^{\pm,\pm}(f)$, $D_j^\pm(f)$ and $E_k^\pm(f)$ denote the corresponding subsets of critical points of $A - D - E$ types. In the set $\mathcal{C}_f^{\text{extr}} := \mathcal{C}_f^{\min} \cup \mathcal{C}_f^{\max}$ of local extremum points, consider the subset $\mathcal{C}_f^{\text{extr}*}$ of degenerate (non-Morse) critical points.

Denote $s := \max\{0, \chi(M) + 1\}$.

Theorem 1. *For any function $f_0 \in C^\infty(M)$, whose all critical points have $A - D - E$ types, the space $\mathcal{F} = \mathcal{F}(f_0)$ has the homotopy type of a manifold $\mathbb{B} = \mathbb{B}(f_0)$ having dimension $\dim \mathbb{B} = 2s + |\mathcal{C}_{f_0}^{\text{extr}}| + |\mathcal{C}_{f_0}^{\text{extr}*}| + 2|\mathcal{C}_{f_0}^{\text{triv}}| + 3|\mathcal{C}_{f_0}^{\text{saddle}}| + 4|\mathcal{C}_{f_0}^{\text{mult}}|$. Moreover:*

- There exists a surjective submersion $\kappa : \mathcal{F} \rightarrow \mathbb{B}$ and a stratification (respectively, a fibration of codimension $|\mathcal{C}_{f_0}|$) on \mathbb{B} such that every $\mathcal{D}^0(\mathbb{R}) \times \mathcal{D}^0(M)$ -orbit (resp., $\mathcal{D}^0(M)$ -orbit) in \mathcal{F} is the κ -preimage of a stratum (resp., a fiber) in \mathbb{B} .*
- The map κ provides a homotopy equivalence between any $\mathcal{D}^0(M)$ -invariant subset $I \subseteq \mathcal{F}$ and its image $\kappa(I) \subseteq \mathbb{B}$. In particular, it provides homotopy equivalences between \mathcal{F} and \mathbb{B} , and between every $\mathcal{D}^0(\mathbb{R}) \times \mathcal{D}^0(M)$ -orbit (resp., $\mathcal{D}^0(M)$ -orbit) from \mathcal{F} and the corresponding stratum (resp., fiber) in \mathbb{B} .*

In particular, $\pi_k(\mathcal{F}) \cong \pi_k(\mathbb{B})$, $H_k(\mathcal{F}) \cong H_k(\mathbb{B})$. Thus $H_k(\mathcal{F}) = 0$ for all $k > \dim \mathbb{B}$.

Our proof of Theorem 1 uses a result (obtained in collaboration with Alexandra Orevkova) about a “uniform” reduction of a smooth function to a normal form near its critical points.

Suppose $\Omega \in \Lambda^n(M)$ is a volume form on a n -manifold $M = M^n$. Let $\mathcal{P} \subset M$ be a finite subset. For any vector field ξ on $M' := M \setminus \mathcal{P}$, we assign the $(n-1)$ -form $\beta = i_\xi \Omega \in \Lambda^{n-1}(M')$. Clearly, this assignment is one-to-one. Furthermore, the flow of the vector field ξ is volume-preserving if and only β is a closed form. Indeed: the Lie derivative $L_\xi \Omega = (i_\xi d + di_\xi)\Omega = di_\xi \Omega = d\beta$, so the

Lie derivative vanishes if and only if $d\beta = 0$. By abusing language, we will call the $(n - 1)$ -form β a *flow*.

Suppose now that $n = \dim M = 2$. A closed 1-form β on $M' = M \setminus \mathcal{P}$ will be called a *gradient-like flow* on M if there exists a Morse function $f \in C^\infty(M)$, called an *energy function* of β , such that

- (i) the set \mathcal{P} coincides with the set of local extremum points of f ,
- (ii) the 3-form $df \wedge \beta|_{M \setminus \mathcal{C}_f}$ has no zeros and defines a positive orientation on M ,
- (iii) in a neighbourhood of every point $P \in \mathcal{C}_f$, there exist local coordinates x, y such that either $\beta = d(xy)$, $f = f(P) + x^2 - y^2$ and $P \in \mathcal{Z} = \mathcal{C}_f \setminus \mathcal{P}$, or $\beta = (xdy - ydx)/(x^2 + y^2)$, $f = f(P) \pm (x^2 + y^2)$ and $P \in \mathcal{P}$.

Geometrically, the set $\mathcal{P}_\beta := \mathcal{P}$ consists of *sources* and *sinks* of the flow β and coincides with the set of local extremum points of the energy function f , while the set $\mathcal{Z}_\beta := \mathcal{Z} = \mathcal{C}_f \setminus \mathcal{P}$ consists of *saddle points* of the flow β and coincides with the set of saddle critical points of f .

Denote by $\mathcal{B}(M)$ the space of all gradient-like flows β on M (having arbitrary finite sets $\mathcal{Z} = \mathcal{Z}_\beta$ and $\mathcal{P} = \mathcal{P}_\beta$ of saddles, sources and sinks depending on β). Endow this space with C^∞ -topology. For a gradient-like flow $\beta_0 \in \mathcal{B}(M)$, denote by $\mathcal{B}(\beta_0)$ the set of all gradient-like flows $\beta \in \mathcal{B}(M)$ having the same local singularities as β_0 (in particular, $|\mathcal{Z}_\beta| = |\mathcal{Z}_{\beta_0}|$ and $|\mathcal{P}_\beta| = |\mathcal{P}_{\beta_0}|$).

We want to describe the topology of the space $\mathcal{B}(\beta_0)$, equipped with the C^∞ -topology, and its decomposition into $\mathcal{D}^0(M)$ -orbits and into classes of (orbital) topological equivalence.

Theorem 2. *For any gradient-like flow β_0 on M , the space $\mathcal{B}(\beta_0)$ has the homotopy type of the manifold $\mathbb{B} = \mathbb{B}(f_0)$ from Theorem 3, where f_0 is an energy function of β_0 . Moreover:*

- (a) *There exists a surjective submersion $\lambda : \mathcal{B}(\beta_0) \rightarrow \mathbb{B}$, a stratification and a $(|\mathcal{P}_{\beta_0}| + |\mathcal{Z}_{\beta_0}|)$ -dimensional fibration on \mathbb{B} such that every class of topological equivalence (resp., every $\mathcal{D}^0(M)$ -orbit) in $\mathcal{B}(\beta_0)$ is the λ -preimage of the stratum (resp., the fibre) from \mathbb{B} .*
- (b) *The map λ provides a homotopy equivalence between every $\mathcal{D}^0(M)$ -invariant subset $I \subseteq \mathcal{B}(\beta_0)$ and its image $\lambda(I) \subseteq \mathbb{B}$. In particular, it provides a homotopy equivalence between $\mathcal{B}(\beta_0)$ and \mathbb{B} , as well as between every class of topological equivalence (resp., every $\mathcal{D}^0(M)$ -orbit) in $\mathcal{B}(\beta_0)$ and the corresponding stratum (resp., fibre) in \mathbb{B} . All fibres and strata in \mathbb{B} (and, thus, all classes of topological equivalence and all $\mathcal{D}^0(M)$ -orbits in $\mathcal{B}(\beta_0)$) are homotopy equivalent either to a point, or to T^2 , or to $SO(3)/G$ or to S^2 , in dependence on whether $\chi(M) < 0$, or $\chi(M) = 0$, or $\chi(M) \cdot |\mathcal{Z}_{\beta_0}| > 0$, or $|\mathcal{Z}_{\beta_0}| = 0$, respectively, where G is a finite subgroup of $SO(3)$.*

In particular, $\pi_k(\mathcal{B}(\beta_0)) \cong \pi_k(\mathbb{B})$, $H_k(\mathcal{B}(\beta_0)) \cong H_k(\mathbb{B})$. Thus $H_k(\mathcal{B}(\beta_0)) = 0$ for all $k > \dim \mathbb{B}$.

We will illustrate our results on several examples.

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REFERENCES

- [1] Elena A Kudryavtseva, Dmitry A Permyakov. Framed Morse functions on surfaces. *Sbornik Math.*, 201(4) : 501–567, 2010.
- [2] Elena A Kudryavtseva. The Topology of Spaces of Morse Functions on Surfaces. *Math. Notes*, 92(2) : 219–236, 2012. <http://arxiv.org/abs/1104.4792>.
- [3] Elena A Kudryavtseva. Special framed Morse functions on surfaces. *Moscow Univ. Math. Bull.*, 67(4) : 14–20, 2012. arXiv:1106.3116.
- [4] Elena A Kudryavtseva. Topology of the spaces of functions with prescribed singularities on surfaces. *Doklady Mathematics*, 93(3) : 264–266, 2016.
- [5] Sergiy I Maksymenko. Homotopy types of stabilizers and orbits of Morse functions on surfaces. *Ann. Glob. Anal. Geom.* 29(3) : 241–285, 2006. arXiv:math.GT/0310067.

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