

International
Scientific Conference



Algebraic
and Geometric
Methods
of Analysis

27-30 May 2024
Odesa, Ukraine

The purpose of this conference is to bring together researchers in geometry, topology, algebra, analysis and dynamical systems and to provide for them a forum to present their recent work to colleagues from different nationalities. This way we aim to stimulate discussion about the latest findings in geometrical and topological methods in analysis and to increase international collaboration.

The conference continues the traditional annual conference «Geometry in Odesa» holding from 2004, and hosted by Odesa National University of Technology (Odesa National Academy of Food Technologies till 2021). From 2017 the conference was renamed to «Algebraic and geometric methods of analysis» (AGMA).

The Conference languages: Ukrainian and English.

LIST OF TOPICS

- Algebraic methods in geometry
- Differential geometry in the large
- Geometry and topology of differentiable manifolds
- General and algebraic topology
- Dynamical systems and their applications
- Geometric and topological methods in natural sciences
- Geometric problems in mathematical analysis

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About square roots of matrices over factorial domains

Volodymyr Prokip

(IAPMM NAS of Ukraine, Naukova Str. 3b, Lviv, 79060)

E-mail: v.prokip@gmail.com

Let R be a factorial domain with identity $e \neq 0$. Notations: $R_{n,m}$ and $R_{n,m}[\lambda]$ are sets of $(n \times m)$ matrices over the domain R and the polynomial ring $R[\lambda]$ respectively, 0_n and I_n are the zero and the identity $n \times n$ matrices respectively, \mathbb{C} is the field of complex numbers, \mathbb{R} is the field of real numbers and \mathbb{Z} is the ring of integers.

It is said that an $n \times n$ matrix B is a square root of the matrix $A \in R_{n,n}$ if $B^2 = A$. The computation of matrix square roots arise in a variety of application domains, including in physics, signal processing, optimal control theory, and many others. The problem of finding square roots from a matrix A over \mathbb{C} or \mathbb{R} is well studied (see [1]–[10] and references therein). Unlike square roots of the complex numbers \mathbb{C} , the square root of a matrix over \mathbb{C} may not exist. For example, the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}_{2,2}$ has no square roots. However, for a nonsingular matrix A over \mathbb{R} there always exists a square root over \mathbb{C} . We note that the existence of a square root of a matrix A over a field depends on the structure of its elementary divisors corresponding to zero eigenvalues. The structure of square roots over a principal ideal domain of the matrix I_n was written in [14].

It is easy to make sure that for nonsingular matrix $A \in R_{n,n}$ does not always have a square root over R . For the matrix $A = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix} \in \mathbb{Z}_{2,2}$ there is no square root over \mathbb{Z} . However, the matrix $B = \begin{bmatrix} i & 0 \\ -i & i \end{bmatrix}$ over the ring of Gaussian integer $\mathbb{Z}[i]$ is the square root of A . In this report we give conditions under which for a matrix $A \in R_{n,n}$ there exists a square root over R .

Let R be a factorial domain. For the matrix $A \in R_{n,n}$ there exists a square root over R if and only if the matrix equation $X^2 = A$ is solvable over R . This equation is solvable if and only if the polynomial matrix $A(\lambda) = I_n \lambda^2 - A$ admits the representation in the form $A(\lambda) = (I_n \lambda - B)(I_n \lambda + B)$, where $B \in R_{n,n}$. From the last equality we have

$$\det A(\lambda) = a(\lambda) = b(\lambda)\tilde{b}(\lambda), \quad (1)$$

where $b(\lambda), \tilde{b}(\lambda) \in R[\lambda]$ – are monic polynomials of degree n . It is evident that condition (1) is the necessary condition for the existence of a square root of the matrix $A \in R_{n,n}$.

For the matrix $A \in R_{n,n}$ and the polynomial $b(\lambda) = \lambda^n + \sum_{i=1}^n b_i \lambda^{n-i} \in R[\lambda]$ (a divisor of the characteristic polynomial of $A(\lambda) = I_n \lambda^2 - A$) we construct the matrices

$$T_A = \begin{bmatrix} \vdots & \vdots \\ O_n & A^3 \\ A^2 & O_n \\ O_n & A^2 \\ A & O_n \\ O_n & A \\ I_n & O_n \\ O_n & I_n \end{bmatrix} \in R_{n(n+1), 2n}, \quad M_b = [I_n \quad I_n b_1 \quad I_n b_2 \quad \dots \quad I_n b_{n-1} \quad I_n b_n] \in R_{n, n(n+1)},$$

$$N_b = [I_n b_1 \quad I_n b_2 + A \quad I_n b_3 \quad \dots \quad I_n b_{n-1} \quad I_n b_n \quad O_n] \in R_{n, (n+1)n}.$$

With matrices T_A , M_b and N_b we associate the $(n \times 2n)$ matrices $M_A = M_b T_A$ and $N_A = N_b T_A$.

In the future, we denote by $d_A(\lambda)$ the g.c.d. minors of $(n - 1)$ -order of the matrix $A(\lambda)$. By virtue of Theorem 1 in [11], we obtain the following statement.

Proposition 1. *Let a matrix $B \in \mathbb{R}_{n,n}$ be a square root of the matrix $A \in \mathbb{R}_{n,n}$, i.e. $B^2 = A$ and $\det(I_n\lambda - B) = b(\lambda)$. If $(b(\lambda), \frac{\det A(\lambda)}{b(\lambda)}, d_A(\lambda)) = e$, then the square root B is uniquely determined by the characteristic polynomial $b(\lambda)$ for the matrix A .*

The proof of the following statements are based on results of papers [12] and [13].

Theorem 2. *Let $A \in \mathbb{R}_{n,n}$ and let $b(\lambda) = \lambda^n + \sum_{i=1}^n b_i \lambda^{n-i} \in \mathbb{R}[\lambda]$ be a divisor of the characteristic polynomial of the matrix $A(\lambda) = I_n\lambda^2 - A$, i.e. $\det A(\lambda) = b(\lambda)\tilde{b}(\lambda)$. If $(b(\lambda), \tilde{b}(\lambda), d_A(\lambda)) = e$, then for matrix A there exists a square root B with characteristic polynomial $b(\lambda) = \det(I_n\lambda - B)$ if and only if the equation $XM_A = N_A$ is solvable. If the square root B exists, then matrix B is uniquely determined by its characteristic polynomial $b(\lambda)$.*

Corollary 3. *Let $A \in \mathbb{R}_{n,n}$ and let $b(\lambda) = \lambda^n + \sum_{i=1}^n b_i \lambda^{n-i} \in \mathbb{R}[\lambda]$ be a divisor of the characteristic polynomial of the matrix $A(\lambda) = I_n\lambda^2 - A$, i.e. $\det A(\lambda) = b(\lambda)\tilde{b}(\lambda)$. If $d_A(\lambda) = \text{const}$, then for matrix A there exists a square root B with characteristic polynomial $b(\lambda) = \det(I_n\lambda - B)$ if and only if the equation $XM_A = N_A$ is solvable. If the square root B exists, then matrix B is uniquely determined by its characteristic polynomial $b(\lambda)$.*

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