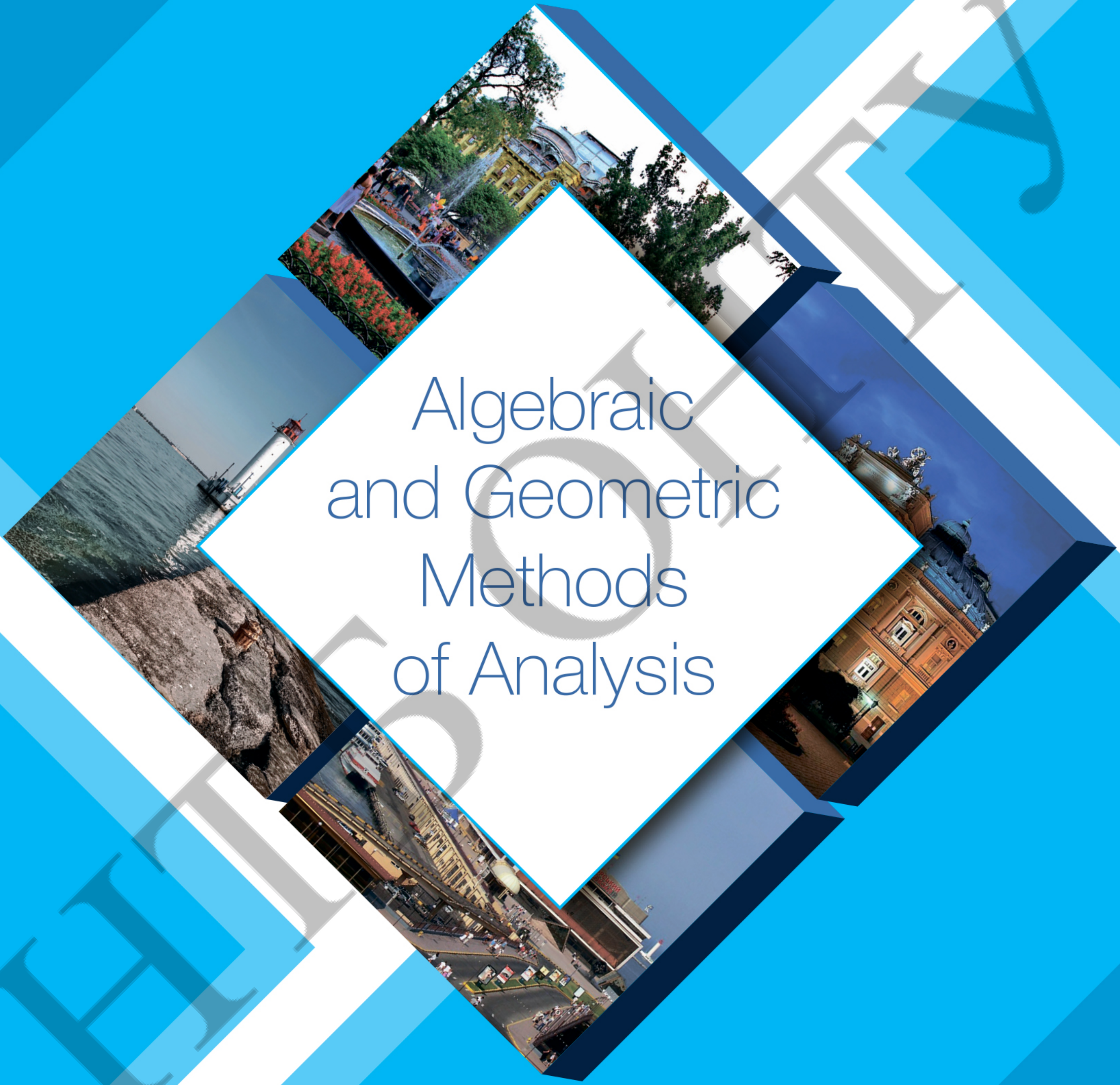
International
Scientific Conference



Algebraic and Geometric Methods of Analysis

27-30 May 2024
Odesa, Ukraine

The purpose of this conference is to bring together researchers in geometry, topology, algebra, analysis and dynamical systems and to provide for them a forum to present their recent work to colleagues from different nationalities. This way we aim to stimulate discussion about the latest findings in geometrical and topological methods in analysis and to increase international collaboration.

The conference continues the traditional annual conference «Geometry in Odesa» holding from 2004, and hosted by Odesa National University of Technology (Odesa National Academy of Food Technologies till 2021). From 2017 the conference was renamed to «Algebraic and geometric methods of analysis» (AGMA).

The Conference languages: Ukrainian and English.

List of topics

- Algebraic methods in geometry
- Differential geometry in the large
- Geometry and topology of differentiable manifolds
- General and algebraic topology
- Dynamical systems and their applications
- Geometric and topological methods in natural sciences
- Geometric problems in mathematical analysis

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Solutions of N-body harmonic oscillators and Calogero-Moser model using Φ^4 matrix model

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The N-body harmonic oscillator system and its generalized system, the Calogero-Moser model, are known as quantum integrable systems, i.e. their Schrödinger equations are solvable and eigenstates of the Hamiltonians can be constructed. It has long been known that there is a connection between these quantum integrable systems and certain kind of generalized matrix models. This talk is concerned with the new correspondence of the quantum solvable systems with some matrix models, which is discovered last year. These matrix models are given as the Grosse-Wulkenhaar models, known as renormalizable scalar Φ^4-theories on Moyal spaces, which are non-commutative spaces. The Moyal space has Fock representation, so field theories can be expressed by using matrix representation. A scalar Φ^4-theory on Moyal space corresponds to a Hermitian Φ^4-matrix model or a real symmetric Φ^4-matrix model. In particular, Φ^4-matrix model known as the Grosse-Wulkenhaar models have kinetic terms $\mathrm{Tr}(E\Phi^2)$, where E is a positive diagonal matrix without degenerate eigenvalues. (These matrix models are also obtained by changing the potential of the Kontsevich model from Φ^3 to Φ^4.) We show that their partition functions of these matrix models correspond to zero-energy solutions of a Schrödinger type equation with the N-body harmonic oscillator Hamiltonian and the Calogero-Moser Hamiltonian, respectively.

Let Φ be a Hermitian or real symmetric $N \times N$ matrix. Let $Z(E,\eta)$ be the partition function defined by

$$Z(E,\eta) = \int d\Phi \; e^{-S_E[\Phi]}, \tag{1}$$

where $S_E = N\ \mathrm{Tr}\{E\Phi^2 + \frac{\eta}{4}\Phi^4\}$ and η is a real number. The domains of integrations are the space of Hermitian $N\times N$-matrices and the space of real symmetric $N\times N$-matrices, respectively. Let $\Delta(E)$ be the Vandermonde determinant $\Delta(E) := \prod_{k<l}(E_l - E_k)$. Using these, the theorem obtained can be described as follows.

Theorem 1. *Let $\Psi(E,\eta)$ be a function defined by*

$$\Psi(E,\eta) := e^{-\frac{N}{\beta\eta}\sum_{i=1}^{N} E_i^2}\Delta(E)^{\frac{\beta}{2}} Z(E,\eta).$$

Then $\Psi(E,\eta)$ is a zero-energy solution of the Schrödinger type equation

$$\mathcal{H}\Psi(E,\eta) = 0.$$

Here $\mathcal{H}$ is the Hamiltonian $\mathcal{H}_{HO}$ for the N-body harmonic oscillator system when we consider the Hermitian matrix model with $\beta = 2$:

$$\mathcal{H}_{HO} := -\frac{\eta}{N}\sum_{i=1}^{N}\left(\frac{\partial}{\partial E_i}\right)^2 + \frac{N}{\eta}\sum_{i=1}^{N}(E_i)^2.$$

When we consider the real symmetric matrix model with $\beta = 1$, then $\mathcal{H}$ is the Hamiltonian $\mathcal{H}_{CM}$ for Calogero-Moser model:

$$\mathcal{H}_{CM} := \frac{-\eta}{2N}\left(\sum_{i=1}^{N}\frac{\partial^2}{\partial E_i^2} + \frac{1}{4}\sum_{i\neq j}\frac{1}{(E_i - E_j)^2}\right) + 2\frac{N}{\eta}\sum_{i=1}^{N} E_i^2. \tag{2}$$

It is known that the N-body harmonic oscillator system or the Calogero-Moser model is associated with a Virasoro algebra structure. Using this fact, families of differential equations satisfied by the partition functions are also obtained from the Virasoro(Witt) algebra representations:

$$[\widetilde{L}_n, \widetilde{L}_m] = (n-m)\widetilde{L}_{n+m}. \tag{3}$$

The definitions of symbols and terms are left to the references [1, 2], but the following theorem is obtained

Theorem 2. *The partition function defined by (1) satisfies*

$$\mathcal{L}_{SD}(\widetilde{L}_{-m}Z(E,\eta)) = -\,2m(\widetilde{L}_{-m}Z(E,\eta)). \tag{4}$$

Here $\mathcal{L}_{SD}$ is a differential operator such that some Schwinger-Dyson equation for the partition function given by

$$\mathcal{L}_{SD}Z(E,\eta) = 0. \tag{5}$$

This means that $\widetilde{L}_{-m}Z(E,\eta)$ is an eigenfunction of $\mathcal{L}_{SD}$ with the eigenvalue $-2m$.

This talk is based on [1], in collaboration with H. Grosse, and [2], in collaboration with H. Grosse, N. Kanomata, and R. Wulkenhaar.

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Partitioning problem and defensive alliances in the context of zero-divisor graphs of rings

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This is joint work with **Driss Bennis**

The partitioning of the vertex set of a graph is a well-studied problem in graph theory. It involves dividing the set of vertices of a graph into disjoint subsets or partitions, based on specific criteria or constraints. In this talk, we are interested in partitioning the zero-divisor graph of a commutative ring into global defensive alliances. This problem has been well investigated in graph theory. Here, we connected it with the ring theoretical context. We characterize various finite commutative rings for which the zero-divisor graph is partitionable into global defensive alliances. We also present several examples to illustrate and delimit the scope of the established results.

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On the connection between algebraic, geometric, and topological methods in the classification of algebraic surfaces and curves

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The classification of algebraic curves and surfaces in a moduli space is a challenging subject in algebraic geometry. Moduli spaces are spaces that parameterize families of algebraic surfaces. They can be used to study the geometry of algebraic surfaces and to compare different surfaces. Classifying algebraic surfaces and curves is an important task because of the comparison between different objects that we study. The moduli space of curves for example, is a space that parameterizes families of algebraic curves of a fixed genus.

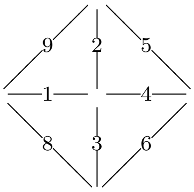
The objects we study and classify can be algebraic curves (via fundamental groups and finding Zariski pairs), algebraic surfaces, and gluing of algebraic surfaces (via deformations and projections).

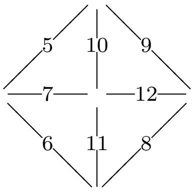
There are methods that can assist in this classification, for example: topological classification, intersection theory, singularities, cohomology, symmetric groups, etc. There are known algorithmic methods as well, and the choice of a method depends on the specific properties of the surface or curve in question and the desired level of detail in the classification.

From the geometric and topological point of view: we consider planar and non-planar deformations and projections to find branch curves of algebraic surfaces. From the algebraic and computational point of view: researchers in the mathematical community use the computer programs Magma, Singular, Maple, and so on. These are just a few examples of software packages that can be used for classifying algebraic surfaces and curves. In our research we use Magma as well, because we investigate fundamental groups and the Magma is a great tool for this goal. We have built some computer softwares to overcome the complicated algebraic computations in the fundamental groups.

Firstly, in order to understand the complexity of the computations in the classification of algebraic surfaces, let us look at the following figure. We can see a high multiplicity of singularities. It happens especially when we glue two planar deformations or when we consider a non-planar deformation. The following figure shows two planar deformations glued together along four edges, and we get a non-planar deformation with multiplicity 4 in all singularities. In this case, the fundamental group of the Galois cover of the surface that has such a deformation is metabelian of order 2^{23} [1].

Now we explain how classification of algebraic surfaces works. We take an algebraic surface embedded in a projective space and project it with a generic projection onto the projective plane. We get the branch curve and then we are able to calculate G - the fundamental group of its complement. A special software gives as an output all braids relating to the branch curve and also

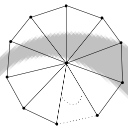




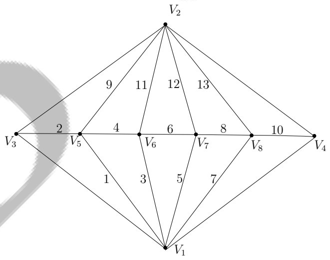
the presentation of G. We then find a certain quotient of G, which is going to be the fundamental group of the Galois cover of the surface. This latter group is an invariant in the classification of algebraic surfaces, and has a geometric significance because it is equal for all the surfaces in the same connected component in the moduli space. We can define an isomorphism between our group and some Coxeter quotient, and we can determine the fundamental groups, using the ideas in [2].

Details about the fundamental groups of Galois cover of an algebraic surface and some interesting examples can be found in our works [3, 4, 5, 6]. In these recent works, we study algebraic surfaces with deformations that have Zappatic R_n singularities (for any n) [3], and also their gluings [4], and surfaces that have non-planar deformations, in which singularities with high complexity appear [6]. Moreover, we study deformations with Zappatic E_n singularities [5].

Theorem 1 ([5]). *The fundamental group of the Galois cover of surfaces that have deformation with one Zappatic E_n singularity is trivial for $n \geq 4$.*



Theorem 2. *Galois covers of a union of two Zappatic surfaces of type R_n are simply-connected surfaces of general type, for any n.*



As for algebraic curves, we can use the software that we constructed in order to produce braids and presentations of fundamental groups and determine these fundamental groups. These computations enable us to get Zariski pairs, see examples in [7, 8]. Moreover, the study of families of curves is an inseparable part of the classification of curves because there we can also calculate invariants, perform deformations, and check how these processes affect the classification [9].

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Nilpotent structures of oriented neutral vector bundles

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Let E be an oriented vector bundle over a manifold M of rank $4n$ and h a neutral metric of E. We call a section N of $\mathrm{End}\,E$ a *nilpotent structure* of E if on a neighborhood of each point of M, there exists an ordered frame field $e = (e_1, \dots, e_{2n}, e_{2n+1}, \dots, e_{4n})$ of E satisfying

$$h(e_i, e_i) = -h(e_{2n+i}, e_{2n+i}) = 1 \ (i = 1, \dots, 2n), \quad h(e_i, e_j) = 0 \ (i \neq j) \tag{1}$$

and $Ne = e\Lambda_n$, where

$$\Lambda_n := \begin{bmatrix} O_n & -I_n & O_n & I_n \\ I_n & O_n & I_n & O_n \\ O_n & I_n & O_n & -I_n \\ I_n & O_n & I_n & O_n \end{bmatrix},$$

I_n is the $n \times n$ unit matrix and O_n is the $n \times n$ zero matrix. Let N be a nilpotent structure of E. We call N an *ε-nilpotent structure* ($\varepsilon \in \{+, -\}$) if on a neighborhood of each point of M, there exists an ordered frame field e giving the orientation of E and satisfying (1) and $NeI'_{4n,\varepsilon} = eI'_{4n,\varepsilon}\Lambda_n$ with

$$I'_{4n,\varepsilon} := \begin{bmatrix} I_n & O_n & O_n & O_n \\ O_n & I_n & O_n & O_n \\ O_n & O_n & I_n & O_n \\ O_n & O_n & O_n & I_{n,\varepsilon} \end{bmatrix}, \qquad I_{1,\pm} := \pm 1, \quad I_{n,\pm} := \begin{bmatrix} \pm 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \ (n \geq 2).$$

Let N be an ε-nilpotent structure of E. Then such a frame field as e is called an *admissible frame field* of N. For an admissible frame field e of N, we set $\xi = \xi_1 \wedge \cdots \wedge \xi_{2n}$, where

$$\begin{array}{ll} \xi_1 := e_1 - e_{2n+1}, & \xi_i := e_i - e_{2n+i}, \\ \xi_{n+1} := e_{n+1} + \varepsilon e_{3n+1}, & \xi_{n+i} := e_{n+i} + e_{3n+i} \end{array} \quad (i = 2, \dots, n).$$

Then ξ does not depend on the choice of an admissible frame field e of N ([3]). Therefore N gives a section ξ_N of the $2n$-fold exterior power $\bigwedge^{2n} E$ of E. A nilpotent structure is characterized by

(i) $\mathrm{Im}\,N = \mathrm{Ker}\,N$, and $\pi_N := \mathrm{Im}\,N = \mathrm{Ker}\,N$ is a light-like subbundle of E of rank $2n$,
(ii) $h(\phi, N\phi) = 0$ for any local section ϕ of E

([2], [3]). In particular, N gives a null structure on each fiber of E and h is null-Hermitian with respect to N (see [9]). The subbundle π_N is locally spanned by $\xi_1, \dots, \xi_{2n}$.

Remark Suppose $n = 1$. Then $\bigwedge^2 E$ is a vector bundle over M of rank 6 and h induces a metric $\hat{h}$ of $\bigwedge^2 E$ of signature (2,4). In addition, $\bigwedge^2 E$ is decomposed as $\bigwedge^2 E = \bigwedge^2_+ E \oplus \bigwedge^2_- E$ by two subbundles $\bigwedge^2_+ E$, $\bigwedge^2_- E$ of rank 3 and the restriction of $\hat{h}$ on each of them has signature (1,2). The *light-like twistor spaces* associated with E are fiber bundles $U_0\big(\bigwedge^2_\pm E\big)$ in $\bigwedge^2_\pm E$ respectively such that each fiber is a light cone. Each light-like line subbundle of $\bigwedge^2_+ E$ or $\bigwedge^2_- E$ corresponds to a light-like subbundle of E of rank 2 and each ε-nilpotent structure N of E corresponds to a section of $U_0\big(\bigwedge^2_\varepsilon E\big)$ given by $(1/\sqrt{2})\xi_N$ ([2], [3]). The space-like twistor spaces $U_+\big(\bigwedge^2_\pm E\big)$ associated with E are fiber bundles in $\bigwedge^2_\pm E$ respectively such that each fiber is a hyperboloid of two sheets. A section of $U_+\big(\bigwedge^2_\varepsilon E\big)$ corresponds to a complex structure of E preserving h. See [1], [5] for the space-like twistor spaces. The time-like twistor spaces $U_-\big(\bigwedge^2_\pm E\big)$ associated with E are fiber bundles in $\bigwedge^2_\pm E$ respectively such that each fiber is a hyperboloid of one sheet. A section of $U_-\big(\bigwedge^2_\varepsilon E\big)$ corresponds to a paracomplex structure of E reversing h. See [1], [13], [14] for the time-like twistor spaces. See [7], [10], [11] for the twistor spaces in the case h is a Riemannian (i.e., positive-definite) metric, which are the prototypes of $U_+\big(\bigwedge^2_\pm E\big)$, $U_-\big(\bigwedge^2_\pm E\big)$ and $U_0\big(\bigwedge^2_\pm E\big)$.

Let ∇ be a connection of E satisfying $\nabla h = 0$. Let N be an ε-nilpotent structure of E. We say that N satisfies the *Walker condition* with respect to ∇ if for any local section ψ of π_N, $\nabla\psi$ is a 1-form valued in π_N. See [6], [9], [16] for Walker manifolds. Let $\hat{\nabla}$ be the connection of $\bigwedge^{2n} E$ induced by ∇. Then N satisfies the Walker condition with respect to ∇ if and only if $\hat{\nabla}\xi_N = \alpha \otimes \xi_N$ for a 1-form α. If $\nabla N = 0$, then $\hat{\nabla}\xi_N = 0$ ([4]) and therefore N satisfies the Walker condition ([9]).

The main objects of research in this talk are special nilpotent structures, and they are called *H-nilpotent structures* of (E, h, ∇), where H is a Lie subgroup of $SO(2n, 2n)$ related to neutral hyperKähler structures. There exist a complex structure I and paracomplex structures J_1, J_2 of E such that h, ∇, I, J_1, J_2 form a neutral hyperKähler structure of E if and only if there exists an H-nilpotent structure of (E, h, ∇) ([4]). See [5], [12] for paraquaternionic structures. See [8], [15] for neutral hyperKähler 4-manifolds.

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Multiplicative b-homogeneralized Derivations of Associative Rings

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In this manuscript, we present multiplicative b-homogeneralized derivation on an associative ring R and discuss certain differential (functional) identities having multiplicative b-homogeneralized derivation. Investigating the centralizer of suitable subset over semiprime rings that admit multiplicative b-homogeneralized derivation enhances some outcomes in the literature. We refer the reader to [4] and [2] for more details.

As is well known, the problem of linear mappings preserving fixed products is a very interesting item in the field of operator algebra. Derivations that can be completely determined by the local action on some subsets of algebra have attracted attention of many researchers. The Martindale ring of quotients of a prime ring R was introduced in [6] as a tool for studying rings satisfying a polynomial identity. The concept was extended to semiprime rings in [5]. Historically, the study of derivation was initiated during the 1950s and 1960s. Derivations of rings got a tremendous development in 1957, when [3] established two very striking results in the case of prime rings.

Named that R is a semiprime when R satisfy the expression $r_1Rr_1 = 0$ which yields $r_1 = 0$ and R is prime if $r_1Rr_2 = 0$ which supply two options there either $r_1 = 0$ or $r_2 = 0$. As a factual information about the connection between the previous concepts a prime and semiprime ring mentioned as following: A prime ring forms another kind of ring, which is a semiprime, while the converse, unfortunately, is not always true.

When a ring R admits for all $r_1, r_2 \in R$ satisfying Leibniz's rule, which is $d(r_1r_2) = d(r_1)r_2 + r_1d(r_2)$ then a derivation is that an additive map $d\colon R \to R$. Whenever for all $r_1, r_2 \in R$ there exists an identity $D(r_1r_2) = D(r_1)r_2 + r_1d(r_2)$. Then, D is an additive mapping defined as $D\colon R \to R$ is recorded as a generalized, *i.e.* a generalized derivation, where d worked as an additive mapping derivation over R.

In 2000, a classical definition of homoderivation posted in El Sofy's article [1], where he was described an additive mapping a homoderivation concerning a ring R like ψ from R to R satisfying $\psi(xy) = \psi(x)\psi(y) + \psi(x)y + x\psi(y)$ where x and y belong to R. Moreover, mapping $F\colon R \to Q_{mr}$ associated with derivation (need not be additive) $d\colon R \to R$ such that $F(\sigma\tau) = F(\sigma)\tau + b\tau d(\tau)$ holds for all $\sigma, \tau \in R$ and any fixed $0 \neq b \in Q_s \subset Q_{mr}$. If F is additive (not necessarily additive), then F is called b-generalized derivation (multiplicative b-generalized.

Definition 1. Suppose that R is an associative ring, mapping $F\colon R \to Q_{mr}$ associated with homoderivation $d\colon R \to R$ such that $F(\sigma\tau) = F(\sigma)F(\tau) + F(\sigma)\tau + b\tau d(\tau)$ holds for all $\sigma, \tau \in R$

and any fixed $0 \neq b \in Q_s \subset Q_{mr}$. When F (is not necessarily additive), then F is called b-homogeneralized derivation (multiplicative b-homogeneralized).

Theorem 2. *Let R be a semiprime ring and K be a nonzero dense ideal of R. Suppose $F\colon R \to Q_{mr}$ is a multiplicative b-homogeneralized derivation associated with derivation $d\colon R \to R$ satisfying the condition $[F(\sigma), \tau] \in Z(R)$ for all $\sigma, \tau \in K$ and any $0 \neq b \in Q_s \subseteq Q_{mr}$.*

Theorem 3. *Let R be a semiprime ring and K be a nonzero dense ideal of R. Assume $F\colon R \to Q_{mr}$ is a multiplicative b-generalized derivation associated with derivation $d\colon R \to R$ such that $F([\sigma, \tau]) = 0$ for all $\sigma, \tau \in K$ and any $0 \neq b \in Q_s \subseteq Q_{mr}$. Then either d is commuting over R is commutative or $[\sigma, b] = 0$.*

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Algebra in fields extended by infinity

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Definition 1. A *corps* is a set F endowed with two binary operations $+, \cdot : F \times F \to F$ and two distinct constants $0, 1 \in F$ that satisfy the following eight axioms:

(1) $\forall x, y, z \in F \ \ \big(x + (y + z) = (x + y) + z\big)$;
(2) $\forall x \in F \ \ (x + 0 = x = 0 + x)$;
(3) $\forall x \in F \ \exists y \in F \ (x + y = 0 = y + x)$;
(4) $\forall x, y, z \in F \ \ (x \cdot (y \cdot z) = (x \cdot y) \cdot z)$;
(5) $\forall x \in F \ (x \cdot 1 = x = 1 \cdot x)$;
(6) $\forall x \in F \setminus \{0\} \ \exists y \in F \ \ (x \cdot y = 1 = y \cdot x)$;
(7) $\forall a, x, y \in F \ \ \big(a \cdot (x + y) = a \cdot x + a \cdot y\big)$;
(8) $\forall x, y, b \in F \ \ \big((x + y) \cdot b = x \cdot b + y \cdot b\big)$.

A corps F is called a *field* if $x \cdot y = y \cdot x$ for all elements $x, y \in F$.

Definition 2. A *procorps*[1] is a set F endowed with two binary operations $+, \cdot : F \times F \to F$ and three distinct constants $0, 1, \infty \in F$ that satisfy the following nine axioms:

(1) $\forall x, z \in F \ \forall y \in F \setminus \{\infty\} \ \ \big(x + (y + z) = (x + y) + z\big)$;
(2) $\forall x, y \in F \ \ (x + y = y + x)$;
(3) $\forall x \in F \ \ (x + 0 = x = 0 + x)$;
(4) $\forall x, z \in F \ \forall y \in F \setminus \{0, \infty\} \ \ (x \cdot (y \cdot z) = (x \cdot y) \cdot z)$;
(5) $\forall x \in F \ (x \cdot 1 = x = 1 \cdot x)$;

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[1] *Procorps* is an abbreviation of a “projective corps”.

(6) $\forall x \in F\ \exists y \in F\ \ (x \cdot y = 1 = y \cdot x)$;
(7) $\forall a \in F \setminus \{0, \infty\}\ \forall x, y \in F\ \ \big(a \cdot (x + y) = a \cdot x + a \cdot y\big)$;
(8) $\forall x, y \in F\ \forall b \in F \setminus \{0, \infty\}\ \ \big((x + y) \cdot b = x \cdot b + y \cdot b\big)$;
(9) $0 \cdot 0 = 0$, $\infty \cdot \infty = \infty$ and $1 + \infty = \infty = \infty + 1$.

A procorp F is called a *profield* if $x \cdot y = y \cdot x$ for all elements $x, y \in F$.

Example 3. Let F be a corps and $\infty \notin F$. Consider the set $\bar{F} := F \cup \{\infty\}$, and extend the operations of addition and multiplication from F to $\bar{F}$ letting

$$\begin{aligned}
&\forall x \in \bar{F} \setminus \{\infty\}\ \ (x + \infty = \infty = \infty + x),\\
&\forall x \in \bar{F} \setminus \{0\}\ \ (x \cdot \infty = \infty = \infty \cdot x),\\
&\infty + \infty = 0, \quad \infty \cdot 0 = 1 = 0 \cdot \infty.
\end{aligned}$$

The set $\bar{F}$ endowed with the extended operations of addition and multiplication and the constants $0, 1, \infty$ is a procorps, called the *projective ∞-extension* of the corps F. If F is a field, then its projective ∞-extension $\bar{F}$ is a profield.

Example 4. The Riemannian sphere $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the projective ∞-extension of the field of complex numbers $\mathbb{C}$.

The following theorem shows that procorps are exactly projective ∞-extensions of corps.

Theorem 5. *For every procorps (profield) $\bar{F}$, the set $F := \bar{F} \setminus \{\infty\}$ endowed with the induced operations of addition and multiplication is a corps (field) and $\bar{F}$ is the projective ∞-extension of F.*

Variational problems in Nonsmooth Analysis

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In the last years, elliptic equations involving a nonsmooth term have attracted several outstanding mathematicians and the interest towards this kind of problems has grown more and more, not only for their intriguing analytical structure, but also in view of their applications in a wide range of contexts. Motivated by this wide interest in the literature, the leading purpose of this talk is to present some recent results on nonsmooth elliptic equations, mainly related to a wide class of functionals defined through multiple integrals of Calculus of Variations. Applications to quasilinear boundary value problems will be presented and some open problems briefly discussed; see [1] and [2, Chapter 8] for related topics.

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On 2-convex embeddings of non-orientable surfaces in four-dimensional Euclidean space

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Let us recall the definition of k-convexity of a subset of a Euclidean space (see [1]).

Definition 1. A subset $C \subset E^n$ of Euclidean space is called k-convex if through each point $x \in E^n \setminus C$ there passes a k-dimensional plane that does not intersect C.

Note that the usual convexity corresponds to case $k = n - 1$.

We present the following result.

Theorem 2. *The Projective plane and the Klein bottle do not admit a 2-convex embedding in a four-dimensional Euclidean space if the embedding is assumed to be* C^2*-smooth or is a PL-embedding such that the valence of the vertices does not exceed five.*

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Mixing optimization in the batch crystallization of CAM

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The citric acid monohydrate (CAM) is an important organic substance but, until 1997, the scientific literature covered mostly the kinetics of nucleation [4] and the crystal growth [5] rather than its production via the crystallization by cooling in a stirred tank reactor (STR). The Department of Chemical Engineering at the University “La Sapienza” of Rome decided to fill that sci-tech gap through a meticulous investigation, with three STRs at the laboratories of San Pietro in Vincoli's district, on the crystallization in discontinuous (batch) of CAM from aqueous solutions. The author participated in that cutting edge experience, as experimenter and coder under the supervision of Prof. Barbara Mazzarotta, in the years 1997-1998 [1]. Our specific tasks were to spot the main operating conditions, to modify them until an *optimal* crystal size distribution (CSD), i.e., large-sized homogeneous crystals of CAM, and to write a QBasic program predicting the outcomes of any test in batch reactors [2]. Here we focus on the influence of the *agitation*, i.e., the role played by the impellers in crystallizing the CAM thanks to their different shapes and speeds. All the data, collected and simulated, show that the three-blade marine impeller performs better than the Rushton turbine and that a low stirring rate gives the best CSD [3]. The homogenous distribution of large crystals from a low agitated round-bottomed tank, stirred via a 3-blade marine impeller, is due to the *optimal* suspension state that the axial flow provides for the dispersed phase of CAM particles [6], as confirmed by the computational fluid-dynamics software `VisiMix`.

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On representation type of incident algebras of extensions of positive posets

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In 1972, P. Gabriel introduced a quadratic form for finite quivers, which was called by him the Tits quadratic form. He proved that a quiver Q is of finite representation type over a field k (i.e., has up to equivalence finitely many indecomposable representations) if and only if its Tits quadratic form is positive. This result laid the foundations of a new direction in the theory of algebras.

In 1974, Yu. A. Drozd showed that a (finite) poset S is of finite representation type if and only if its quadratic Tits form

$$q_S(z) = z_0^2 + \sum_{i\in S} z_i^2 + \sum_{i<j,i,j\in S} z_i z_j - z_0 \sum_{i\in S} z_i$$

is weakly positive, i.e., positive on the non-zero vectors with non-negative coordinates (representations of posets were introduced by L. A. Nazarova and A. V. Roiter in 1972). In contrast to quivers, the posets with weakly positive and with positive Tits forms do not coincide. Since the connected quivers having positive Tits quadratic form coincide with the quivers whose underlying graphs are (simply faced) Dynkin diagrams, the posets with positive Tits form are analogs of the Dynkin diagrams. Such posets, which are simply called positive, were classified by the authors in [1]. Up to isomorphism and duality, the positive posets consist of 5 series and 108 discrete ones.

Since the incidence algebras of positive posets are of finite representation type, it is natural to study representation type of their element-extensions. We consider cases when "new elements" are nodes (in the sence that they are comparable with all other elements). Here we formulate some consequences of our investigation.

A positive poset T is called serial if there exists infinite consequence $T \subset T_1 \subset T_2 \subset \dots$ such that all posets T_i are also positive. By definition, the incidence algebra $\mathcal{I}_k(T)$ of a poset T of order n is the matrix algebra which consisits of all matrices $M = (m_{ij})$, $i, i \in T$ (over a field k) such that $m_{ij} = 0$ if $i > j$. Finally, we write $X > Y$ for subposets X, Y of a poset T if $x > y$ for any $x \in X, y \in Y$.

Theorem 1. *Let S be a positive poset and $\overline{S} = S \cup X$ be an extension of S with chained $X > S$. Then the incidence algebra $\mathcal{I}_k(\overline{S})$ is of infinite representation type if the length $l(X)$ of X is greater than 5.*

Theorem 2. *Let $\overline{S}$ be as in Theorem 1 but S is non-serial. Then the incidence algebra $\mathcal{I}_k(\overline{S})$ is of infinite representation type if*
(a) the width of S is equal to 2 and $l(X) > 4$;
(b) the width of S is equal to 3 and $l(X) > 1$.

In all statements the estimates are exact.

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A theorem on hypercohomology groups and singular homology in field theory

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In this research we consider a theorem that relates the hypercohomology groups obtained with the spectrum through the its singular homology taking components $\mathbb{Z}_{tr}(k)$ and the A_1 -homotopy in the action of the symmetric $\mathrm{DM}^{eff,-}_{Nis}(k)$. Then we can characterize in a projective vector bundle the solutions of the field equations d$da = 0$ on singularities.

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Geometric and algebraic-topological structures in Schwartz distribution spaces for relativistic Quantum Mechanics

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The classic Hilbert space methods cannot be used for the definition and resolution of the free relativistic Schrödinger equation because the very fundamental solutions of this equation cannot be framed in a Hilbert or Banach space context. We can justify the necessity to use distribution theory, for many reasons. If we lock down ourselves in separable Hilbert space theory, we cannot hope to solve satisfactorily (from a physical point of view) the relativistic Schrödinger equation for free particles. The simple reason is that the very main (and generating) solutions of the relativistic Schrödinger equation for free particles cannot be considered as elements of a separable Hilbert Space. First of all, if we desire to consider the standard L^2 product, we immediately observe that the de Broglie waves do not belong to the space of square integrable functions. Moreover, the set of all harmonic waves is a continuous set (not a discrete one) and, if we select its "naturally orthonormal" subfamily (that generating the unitary Minkowsky-Fourier transform as integral kernel), we are getting again a continuous family that should be orthogonal by right, from a physical perspective, but cannot be as such in any separable Hilbert space (even different from L^2). We cannot find continuous orthonormal families in a separable Hilbert space, but only discrete orthonormal systems! Furthermore, even forcing the matter and considering a Hilbert space generated by all those "unitary orthogonal waves", we would obtain a non-separable Hilbert space, that would complicate enormously the matter, from a functional calculus point of view, because we have no reasonable or natural spectral theory for non-separable Hilbert spaces. We are not saying, here, that we have not to use nonseparable Hilbert spaces in Quantum mechanics, but we see that they do not help in the formulation and resolution of the relativistic Schrödinger equation.

Analogous problems we would risk to face if we lock down ourselves in separable Normed Space theory: it is very hard to keep, in a unique functional theory, a reasonable and convenient separable norm with a good spectral theory and the presence of the continuous family of de Broglie waves. That is a general problem, in quantum mechanics and quantum field theory: when we consider harmonic waves and related differential equations (or operatorial equations), theoretical phisicists, actually, do not use Hilbert Space theory - and we know it - we use, instead, smooth function theory, differentiable function theory, we work, essentially, with calculus techniques and distribution theory. Then, we have another general problem of Hilbert space theory in QM: even when we solve the classic Dirac free equation, the manipulations and resolutions proceed, essentially, in a differentiable theory context; indeed, the basis solutions of the free Dirac equation are bispinors constructed by harmonic waves and then, automatically, we work out of the Hilbert space theory. Moreover, when we work out of the Hilbert space theory, we work also out of the spectral theory on Hilbert spaces. Consequently, we cannot expect to find a correct and unambiguous definition of the square root of an operator by Hilbert space techniques, if the domain of such operator should contain the de broglie waves, because in this case we are playing outside of any separable Hilbert space. It's not a case (and do not surprise at all) that Dirac equation was solved by smooth calculus and finite algebraic methods, rather than infinite dimensional Hilbert space techniques. In order to define the square-roots of differential operators, in quantum mechanical context (where we need to manage harmonic waves, eigenstates of position operator, continuous spectra, and so on...), we need a spectral theory constructed elsewhere, not in Hilbert spaces. In some way, quantum mechanics needs to coordinate and put together two apparently incompatible aspects: the state space of a quantum system can be generated (simultaneously) by discrete and continuous bases, at the same time: the position and momentum basis ($|x>$), ($|p>$) are continuous, while the Hermite function basis is discrete. Very often we read "let's solve the harmonic oscillator problem in the position basis", or "let's solve the harmonic oscillator problem in the momentum basis", which are continuous basis (in some Radon-Pettis-Schwartz sense to be correctly defined) only to see, after a while, that the harmonic oscillator is solved by the discrete Hermite function basis of L^2 (it would be better to say of the Schwartz function Space $\mathcal{S}$). How the position basis and momentum basis (that completely stay out of L^2) can generate the same state space generated by the L^2 Hermite function basis? In what sense a continuous family of vectors can generate a functional space? The position eigenstates are not even functions, they are measures. In what space are we moving? The state space is separable or not? What is its Hilbert dimension, $\aleph_0$ or $\aleph_1$? How a separable Hilbert (or Banach) space could contain "non-normalizable" vectors and continuous orthogonal families of non-normalizable vectors that, from a physical point of view, represent simply the certainty to observe a specific result? In tempered distribution spaces, we know that the Hermite function family is a discrete basis in a rightful algebraic-topological sense, it is a total family, it is also a basis in a generalized Hilbert sense (with respect to the tempered distribution topology). Moreover, position and momentum basis lives in $\mathcal{S}'$ and generate $\mathcal{S}'$ in the Schwartz linear algebraic sense S' is a separable topological vector space, it is wonderfully generated by discrete and continuous basis, in two different rigorous and operative meanings and, by the way, exactly the meaning used practically by quantum physicists, in a more heuristic way. In addition to that, in the distribution approach, any quantum mechanics observable is a continuous and everywhere defined operator, while in the Hilbert space approach we almost surely face discontinuous (unbounded) operators and very strange, unnatural domains, even for the most simple observables (position, momentum, ladder operators, number operator and so on and so forth). Consequently, in Hilbert spaces, we face any kind of difficulties, even to add or multiply two straightforward operators such as a derivative operator and the position operator - which show different domains - that without, and well before, coming to ask "what the principal square root of a discontinuous, non-everywhere defined, not-properly hermitian, densely defined (or perhaps closable) operator is". What our new Schwartz linear Algebra theory clarifies compared to previous methods? The new theory clarifies definitely where we are working, in quantum theories and quantum field theory, especially in relativistic quantum mechanics. We clarify that the state spaces of quantum objects are not Hilbert spaces (if we exclude the finite dimensional spaces that, anyway, are subspaces of the tempered distribution space $\mathcal{S}'$), but powers of tempered distribution spaces. Fortunately, tempered distribution spaces contain a lot of good inner product spaces, useful in the evaluation of probabilities and expectation values for quantum mechanics. By those inner products, we finally understand the possible way to "normalize" properly the eigenvectors of the position and momentum operators (eigenstates that should be normalizable because of their straightforward physical meaning: they simply represent certainty). First of all, we solve the problem of evolution in the tempered distribution space. Then, when necessary, we go to find the right inner product subspace, of tempered distribution space, in which calculate probabilities and expectation values.

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Lie subalgebras of real order-three special linear Lie algebra revisited

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The study of subalgebras within both real and complex Lie algebras presents a complex challenge, which arises in many fields of mathematics and its applications. For instance, listing inequivalent subalgebras of the maximal Lie invariance algebra of a system of differential equations could result in constructing its "inequivalent" exact solutions. These classifications also serve as an efficient tool in the realms of theoretical physics and the study of integrable systems. At the same time, they themselves remain to be interesting algebraic problems.

In [1] we review the entire framework of the subalgebra classification problem following [2] and references therein, and also suggest new points of view on these methods and rigorously present their theoretical framework. We apply the developed enhanced methods for refining the classification of subalgebras of the Lie algebra $\mathfrak{sl}_3(\mathbb{R})$ and as a byproduct we first obtain the complete classification of the subalgebras of real rank-two affine Lie algebra $\mathfrak{aff}_2(\mathbb{R})$. The real order-three special linear Lie algebra $\mathfrak{sl}_3(\mathbb{R})$ is the algebra of traceless 3×3 matrices with the standard matrix commutator as the Lie bracket and it is spanned by the matrices

$$E_1 := \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},\quad E_2 := \frac{1}{2}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\quad E_3 := \begin{pmatrix}0&-1&0\\0&0&0\\0&0&0\end{pmatrix},\quad D := \frac{1}{6}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix},$$

$$P_1 = \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\quad P_2 = \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\quad R_1 = \begin{pmatrix}0&0&0\\0&0&0\\0&-1&0\end{pmatrix},\quad R_2 = \begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}.$$

In this way, the algebra $\mathfrak{sl}_3(\mathbb{R})$ is defined through its faithful irreducible representation of the minimal dimension, which is exactly the vector space $\mathbb{R}^3$.

The best attempt in listing inequivalent subalgebras of $\mathfrak{sl}_3(\mathbb{R})$ was carried out in [2], however it contains a number of misprints, mistakes and the major result is presented without proof. The analogous subalgebra classification for the Lie algebras $\mathfrak{sl}_n(\mathbb{R})$, $n \geqslant 4$, remains to be a significant open problem.

To classify Lie algebras of a simple Lie algebra $\mathfrak{sl}_3(\mathbb{R})$ modulo $\mathrm{SL}_3(\mathbb{R})$-equivalence we adopt the approach detailed in [1, Section 2.1]. Specifically for $\mathfrak{sl}_3(\mathbb{R})$ we go through the following steps:

(*i*) using the defining representation $\mathbb{R}^3$ of the Lie algebra $\mathfrak{sl}_3(\mathbb{R})$ determine all its maximal reducibly and irreducibly embedded subalgebras;

(*ii*) for each of the identified maximal subalgebras construct the lists of inequivalent subalgebras with respect to the action of their corresponding inner automorphism groups;

(*iii*) merge the obtained lists modulo the action of the group $\mathrm{SL}_3(\mathbb{R})$.

This analysis reveal that the Lie algebra $\mathfrak{sl}_3(\mathbb{R})$ contains two irreducibly embedded maximal subalgebras: the special orthogonal Lie algebras $\mathfrak{so}_3(\mathbb{R})$ and $\mathfrak{so}_{2,1}(\mathbb{R})$, as well as two reducibly embedded maximal subalgebras $\mathfrak{a}_1 = \langle E_1, E_2, E_3, E_4, D, P_1, P_2\rangle$ and $\mathfrak{a}_2 = \langle E_1, E_2, E_3, E_4, D, R_1, R_2\rangle$. Both of the latter subalgebras are isomorphic to the rank-two affine Lie algebra $\mathfrak{aff}_2(\mathbb{R})$. According to the step (*ii*), the classification of the subalgebras of $\mathfrak{aff}_2(\mathbb{R})$ is an essential step in the course of addressing the primary problem.

The Lie algebra $\mathfrak{aff}_2(\mathbb{R})$ is the semidirect product $\mathfrak{gl}_2(\mathbb{R}) \ltimes \mathbb{R}^2$. Therefore, to classify subalgebras of $\mathfrak{aff}_2(\mathbb{R})$ we apply the approach for classifying Lie subalgebras of the semidirect products from [1, Section 2.3] to the Lie algebra $\mathfrak{gl}_2(\mathbb{R}) \ltimes \mathbb{R}^2$. Consequently, we present for the first time the complete list of inequivalent subalgebras of the rank-two affine Lie algebra $\mathfrak{aff}_2(\mathbb{R})$ in [1, Theorem 11]. In fact, the classification of subalgebras of the algebra $\mathfrak{aff}_2(\mathbb{R})$ was initiated in [2, Section 3.3], where its inequivalent "twisted" and "nontwisted" subalgebras were listed, however this classification were not completed. Moreover, the validity of these lists is questionable, since to construct them it is essential to have the correct classification of subalgebras of $\mathfrak{gl}_2(\mathbb{R})$, which in [2, eq. (3.11)] was presented with a mistake and a number of misprints. This was an additional motivation for us to thoroughly and comprehensively classify the subalgebras of $\mathfrak{aff}_2(\mathbb{R})$.

To combine the derived lists of inequivalent subalgebras $\mathfrak{a}_1$ and $\mathfrak{a}_2$ modulo the $\mathrm{SL}_3(\mathbb{R})$-equivalence, we specify the following general proposition to the case of the Lie algebra $\mathfrak{sl}_3(\mathbb{R})$.

Proposition 1. *Let $\mathfrak{m} \subset \mathfrak{g}$ be a Lie subalgebra, $M \subset G$ the corresponding Lie subgroup. Choose some subset $C \subset G$ such that $MC = G$. Then Lie subalgebras $\mathfrak{h}_1 \subset \mathfrak{m}$ and $\mathfrak{h}_2 \subset \mathfrak{g}$ are conjugated if and only if there exists an element $g \in C$ such that $\mathrm{Ad}_g\mathfrak{h}_2 \subset \mathfrak{m}$ and moreover $\mathrm{Ad}_g\mathfrak{h}_2$ is equivalent to $\mathfrak{h}_1$ up to $\mathrm{Inn}(\mathfrak{m})$-equivalence.*

To this end, the merging procedure results in the next complete list of irredundant representatives of equivalence classes of subalgebras.

Theorem 2. *A complete list of proper $\mathrm{SL}_3(\mathbb{R})$-inequivalent subalgebras of the algebra $\mathfrak{sl}_3(\mathbb{R})$ is exhausted by the subalgebras, where $\varepsilon \in \{-1, 1\}$, $\delta \in \{0, 1\}$, $\kappa \geqslant 0$, $\mu \in [-1, 3]$, $\mu' \in [0, 1]$ and $\gamma \in \mathbb{R}$:*

$$
\begin{aligned}
&1\mathrm{D}\colon\ \mathfrak{f}^{\delta}_{1.1} = \langle E_1 + \delta P_1\rangle,\quad \mathfrak{f}^{\gamma}_{1.2} = \langle E_1 + E_3 + \gamma D\rangle,\quad \mathfrak{f}^{\mu'}_{1.3} = \langle E_2 + \mu' D\rangle,\quad \mathfrak{f}_{1.4} = \langle E_1 + D\rangle,\\
&2\mathrm{D}\colon\ \mathfrak{f}_{2.1} = \langle P_1, P_2\rangle,\ \mathfrak{f}^{\delta}_{2.2} = \langle E_1 + \delta P_1, P_2\rangle,\ \mathfrak{f}_{2.3} = \langle E_2 + D + P_2, P_1\rangle,\ \mathfrak{f}_{2.4} = \langle E_1 + D, P_2\rangle,\\
&\qquad \mathfrak{f}_{2.5} = \langle E_1, D\rangle, \mathfrak{f}_{2.6} = \langle E_2, D\rangle, \mathfrak{f}_{2.7} = \langle E_1 + E_3, D\rangle,\\
&\mathfrak{f}^{\gamma}_{2.8} = \langle E_2 + \gamma D, E_1\rangle, \mathfrak{f}_{2.9} = \langle E_2 - 3D, E_1 + P_1\rangle,\\
&3\mathrm{D}\colon\ \mathfrak{f}_{3.1} = \langle E_1, P_1, P_2\rangle,\quad \mathfrak{f}_{3.2} = \langle D, P_1, P_2\rangle,\quad \mathfrak{f}^{\kappa}_{3.3} = \langle E_2 + \kappa D, P_1, P_2\rangle,\\
&\qquad \mathfrak{f}^{\varepsilon}_{3.4} = \langle E_1 + \varepsilon D, P_1, P_2\rangle,\ \mathfrak{f}^{\gamma}_{3.5} = \langle E_1 + E_3 + \gamma D, P_1, P_2\rangle,\ \mathfrak{f}^{\gamma}_{3.6} = \langle E_1 + E_3 + \gamma D, R_1, R_2\rangle,\\
&\qquad \mathfrak{f}^{\mu}_{3.7} = \langle E_2 + \mu D, E_1, P_2\rangle,\quad \mathfrak{f}_{3.8} = \langle E_2 - 3D, E_1 + P_1, P_2\rangle,\quad \mathfrak{f}_{3.9} = \langle E_1, E_2, D\rangle,\\
&\qquad \mathfrak{f}_{3.10} = \langle E_1, E_2, E_3\rangle, \mathfrak{f}_{3.11} = \langle E_1 + E_3, P_1 - R_2, P_2 + R_1\rangle,\\
&\mathfrak{f}_{3.12} = \langle E_1 + E_3, P_1 + R_2, P_2 - R_1\rangle,\\
&4\mathrm{D}\colon\ \mathfrak{f}_{4.1} = \langle E_1, D, P_1, P_2\rangle,\quad \mathfrak{f}_{4.2} = \langle E_2, D, P_1, P_2\rangle,\quad \mathfrak{f}_{4.3} = \langle E_1 + E_3, D, P_1, P_2\rangle,\\
&\qquad \mathfrak{f}^{\gamma}_{4.4} = \langle E_2 + \gamma D, E_1, P_1, P_2\rangle,\quad \mathfrak{f}_{4.5} = \langle E_1, E_2, D, P_2\rangle,\quad \mathfrak{f}_{4.6} = \langle E_1 + E_3, D, R_1, R_2\rangle,\\
&\qquad \mathfrak{f}_{4.7} = \langle E_1, E_2, E_3, D\rangle,\\
&5\mathrm{D}\colon\ \mathfrak{f}_{5.1} = \langle E_1, E_2, D, P_1, P_2\rangle,\quad \mathfrak{f}_{5.2} = \langle E_1, E_2, E_3, P_1, P_2\rangle,\quad \mathfrak{f}_{5.3} = \langle E_1, E_2, E_3, R_1, R_2\rangle,
\end{aligned}
$$

$$6D\colon\ \mathfrak{f}_{6.1} = \langle E_1, E_2, E_3, D, P_1, P_2\rangle, \quad \mathfrak{f}_{6.2} = \langle E_1, E_2, E_3, D, R_1, R_2\rangle.$$

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Conformal mappings and a non-holonomic frame

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It is convenient to use a holonomic coordinate systems and associated the so-called natural frame if one explore conformal mappings of differentiable manifolds. But to study some physical applications, in particular spinor fields, we have to use non-holonomic frame which sometimes referred to as the vielbein formulation [1, 2] . In this formulation we introduce 4 independent vectors $t^i_a(x)$, $(a = 0, 1, 2, 3)$ at each point of a spacetime $(V^{1,3}, g)$, which are orthogonal to each other and have a unit length:

$$t^i_a(x) t^j_b(x) g_{ij}(x) = \eta_{ab}, \quad \eta_{ab} = diag(1, -1, -1, -1).$$

Also there exists the inverse matrix t^a_i, which satisfies

$$t^i_a(x) t^a_j(x) = \delta^i_j, \quad t^i_a(x) t^b_i(x) = \delta^a_b.$$

The field $t^a_i(x)$ is called the vielbein [1]. Using such non-holonomic coordinate systems, we should introduce the spin connection by the relation below

$$\omega_k{}^a{}_b = (t^i_b \Gamma^h_{ki} + \partial_k t^h_b) t^a_h. \tag{1}$$

From (1) it follows that

$$\partial_k t^h_a + \Gamma^h_{jk} t^j_a - \omega_k{}^b{}_a t^h_b = 0.$$

The covariant derivative of a spinor field $\psi(x)$ one calculates using the formula:

$$\nabla_k \psi = \partial_k \psi - \frac{1}{4} \omega_{kab} \gamma^{ab} \psi = \partial_k \psi + \Gamma_k \psi,$$

where $\gamma^{ab} = \frac{1}{2}(\gamma^a \gamma^b - \gamma^b \gamma^a)$ is the antisymmetrized product of two gamma matrices. For the adjoint spinor $\overline{\psi} = \psi^\dagger \gamma^0$ we have

$$\nabla_k \overline{\psi} = \partial_k \overline{\psi} + \overline{\psi} \frac{1}{4} \omega_{kab} \gamma^{ab} = \partial_k \overline{\psi} - \overline{\psi} \Gamma_k,$$

If we consider a conformal mapping $f : (V^{1,3}, g) \to (\tilde{V}^{1,3}, \tilde{g})$, i.e. $\tilde{g}_{ij} = e^{2\varphi(x)} g_{ij}$, then the vielbein is transformed as

$$\tilde{t}_i^a(x) = e^{\varphi(x)} t_i^a(x). \tag{2}$$

Here $\varphi(x)$ is some function. Under a conformal transformation the spin connection transforms as

$$\tilde{\omega}_{kab} = \omega_{kab} + t_{ka}\varphi_b - t_{kb}\varphi_a.$$

Here $\varphi_b = \partial_b \varphi = t_b^j \partial_j \varphi$. Hence for the spin-affine connection Γ_k we get:

$$\tilde{\Gamma}_k = \Gamma_k - \frac{1}{4}(t_{ka}\varphi_b - t_{kb}\varphi_a)\gamma^{ab} = \Gamma_k - \frac{1}{2} t_{ka}\varphi_b \gamma^{ab}. \tag{3}$$

On the other hand, the stress-energy tensor for the spinor field ($s = \frac{1}{2}$) in a spacetime $(V^{1,3}, g)$ we could calculate by the formula [3]:

$$T_{jk} = \frac{i}{2}\big(\overline{\psi}\gamma_{(j}\nabla_{k)}\psi - (\nabla_{(j}\overline{\psi})\gamma_{k)}\psi\big), \tag{4}$$

where $\gamma_j = \gamma_a t_j^a(x)$. Taking into account (2), (3), (4) we obtain the transformed stress-energy tensor:

$$\tilde{T}_{jk} = e^{\varphi(x)}\big(T_{jk} - \frac{i}{4}\big(\overline{\psi}\gamma_j t_{ka}\varphi_b\gamma^{ab}\psi + \overline{\psi}\gamma_k t_{ja}\varphi_b\gamma^{ab} + \overline{\psi} t_{ka}\varphi_b\gamma^{ab}\gamma_j + \overline{\psi} t_{ja}\varphi_b\gamma^{ab}\gamma_k\psi\big)\big),$$

However we have the scalar which is preserved under conformal mappings:

$$|A|^2 = A^i g_{ij} A^j = \overline{\psi}\gamma^i\psi g_{ij}\overline{\psi}\gamma^j\psi,$$

where $A^i = \overline{\psi}\gamma^i\psi$ is so called four-dimensional current of the spinor field ψ.

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On topologization of the bicyclic monoid

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In this paper we shall follow the terminology of [2, 4, 5, 6].

A semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an inverse semigroup, then the function $\mathrm{inv}\colon S \to S$ which assigns to every element x of S its inverse element x^{-1} is called the *inversion*. On an inverse semigroup S the semigroup operation determines the *natural partial order* $\preccurlyeq$ on S: $s \preccurlyeq t$ if and only if there exists $e \in E(S)$ such that $s = te$.

A topology τ on a semigroup S is called:

- a *semigroup* (*shift-continuous*) topology if (S, τ) is a topological (semitopological) semigroup;
- an *inverse semigroup* topology if (S, τ) is a topological inverse semigroup;

- an *inverse shift-continuous* topology if (S,τ) is a semitopological semigroup with continuous inversion.

The bicyclic monoid $\mathcal{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq=1$. The bicyclic monoid admits only the discrete semigroup Hausdorff topology [3]. Bertman and West in [1] extended this result for the case of Hausdorff semitopological semigroups.

We construct two non-discrete inverse semigroup T_1-topologies and a compact inverse shift-continuous T_1-topology on the bicyclic monoid $\mathcal{C}(p,q)$. Also we give conditions on a T_1-topology τ on $\mathcal{C}(p,q)$ to be discrete.

Theorem 1. *Every shift-continuous Baire T_1-topology τ on the bicyclic monoid $\mathcal{C}(p,q)$ is discrete.*

Theorem 2. *Let τ be an inverse semigroup T_1-topology on $\mathcal{C}(p,q)$. If there exists a point $q^ip^j\in\mathcal{C}(p,q)$ such that the space $\updownarrow_{\preccurlyeq}q^ip^j$ is quasi-regular at q^ip^j, then τ is discrete.*

Theorem 3. *Let τ be a shift-continuous T_1-topology on the bicyclic monoid $\mathcal{C}(p,q)$ such that the maps $\mathcal{C}(p,q)\to E(\mathcal{C}(p,q))$, $x\mapsto xx^{-1}$ and $\mathcal{C}(p,q)\to E(\mathcal{C}(p,q))$, $x\mapsto x^{-1}x$ are continuous. If there exists a point $q^ip^j\in\mathcal{C}(p,q)$ such that the space $\updownarrow_{\preccurlyeq}q^ip^j$ is semiregular at q^ip^j, then τ is discrete.*

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An Application to Sasaki Extremal metrics via the Berglund-Hübsch rule

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Sasaki-extremal metrics were introduced in [1] as a generalization of metrics with constant scalar curvature, which is obstructed by the Futaki invariant. On this talk we exhibit examples of homotopy spheres and rational homology spheres realized as links of chain-cycle polynomials that do not admit Sasaki extremal metrics in the whole Sasaki cone, which has dimension greater than one. For this, we consider links that are given as 2-fold branched covers of S^9 whose branching loci are rational homology 7-spheres which are links of certain invertible polynomials of chain-cycle type studied in [5] and later in [4] through the Berglund-Hübsch rule of classical mirror symmetry. In [2], Boyer and van Coevering defined a relative version of the K-stability of Collins and Székelyhidy [6, 7] and obtain the first examples of Sasaki manifolds with Sasaki cone of dimension greater than one not admitting Sasaki extremal metrics in the whole Sasaki cone. Based on their result we exhibit 37 new families of links with Sasaki cone of dimension 2 such the whole cone does not admit any extremal representative. All the examples produced here are eitheir homotopy 9-spheres, rational homology 9-spheres or manifolds of the form $S^4 \times S^5$. We can easily extrapolate these examples to arbitrary dimensions where the corresponding Sasaki cones have larger dimensions. All these examples are inequivalent to the ones found in [2]. These examples are consequences of the following more general result that we present in this talk:

Proposition 1 ([5]). *Consider a polynomial of chain-cycle type of the form*

$$f = z_0^{a_0} + z_0 z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}$$

with $a_1 = 2$ *whose link* L_f *is a rational homology sphere and that cuts out a projective hypersurface in* $\mathbb{P}(w_0, w_1, \ldots, w_4)$ *such that*

$$(w_0, w_1, w_2, w_3, w_4) = (m_3 v_0, m_3 v_1, m_2 v_2, m_2 v_3, m_2 v_4),$$

with m_3 *odd,* $\gcd(m_2, m_3) = 1$ *and* $d = m_3 m_2$*. The polynomial*

$$g = f^T + z_5^2 + \ldots + z_n^2$$

with f^T *the Berglund-Hübsch transpose of* f*, determines a link* L_g *such that*

1. *If* n *is even, then* L_g *is a rational homology* $(2n-1)$*-sphere.*
2. *If* n *is odd, then* L_g *is a homotopy* $(2n-1)$*-sphere and* $\Delta_g(-1) = m_3$*. In particular the diffeomorphism type of* L_g *is determined by* m_3*.*
3. *The Sasaki cone of* L_g *has dimension* $1 + \lfloor \frac{n-3}{2} \rfloor$ *and there are no extremal Sasaki metrics in the Sasaki cone of the link* L_g*.*
4. *If* m_3 *is even, then for* $n = 5$*, the link* L_g *has the form* $S^4 \times S^5$*.*

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On singularities of mappings with a Lebesgue integrable majorant

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The following definitions are from [1]. A path γ in $\mathbb{R}^n$ is a continuous mapping $\gamma : \Delta \to \mathbb{R}^n$ where Δ is an interval in $\mathbb{R}$. Its locus $\gamma(\Delta)$ is denoted by $|\gamma|$. Given a family Γ of paths γ in $\mathbb{R}^n$, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for Γ, abbr. $\rho \in \operatorname{adm}\Gamma$, if

$$\int\limits_{\gamma} \rho(x)|dx| \geqslant 1$$

for each (locally rectifiable) $\gamma \in \Gamma$. Given $p \geqslant 1$, the *p-modulus* of Γ is defined by the relation

$$M_p(\Gamma) := \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^p(x) dm(x) \tag{1}$$

interpreted as $+\infty$ if $\operatorname{adm}\Gamma = \varnothing$.

Given sets E and F and a given domain D in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, we denote by $\Gamma(E, F, D)$ the family of all paths $\gamma : [0, 1] \to \overline{\mathbb{R}^n}$ joining E and F in D, that is, $\gamma(0) \in E$, $\gamma(1) \in F$ and $\gamma(t) \in D$ for all $t \in (0, 1)$. Everywhere below, unless otherwise stated, the boundary and the closure of a set are understood in the sense of the extended Euclidean space $\overline{\mathbb{R}^n}$. Let $x_0 \in \overline{D}$, $x_0 \neq \infty$,

$$S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}, S_i = S(x_0, r_i)\,, \quad i = 1, 2\,,$$

$$A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}\,.$$

Let $f : D \to \mathbb{R}^n$, $n \geqslant 2$, and let $Q : \mathbb{R}^n \to [0, \infty]$ be a Lebesgue measurable function such that $Q(y) \equiv 0$ for $y \in \mathbb{R}^n \setminus f(D)$. Let $A = A(y_0, r_1, r_2)$ and $\Gamma_f(y_0, r_1, r_2)$ denotes the family of all paths $\gamma : [a, b] \to D$ such that $f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$, i.e., $f(\gamma(a)) \in S(y_0, r_1)$, $f(\gamma(b)) \in S(y_0, r_2)$, and $f(\gamma(t)) \in A(y_0, r_1, r_2)$ for any $a < t < b$. We say that f *satisfies the inverse Poletsky inequality at* $y_0 \in f(D)$ *with respect to p-modulus,* if the relation

$$M_p(\Gamma_f(y_0, r_1, r_2)) \leqslant \int\limits_{A} Q(y) \cdot \eta^p(|y - y_0|)\, dm(y) \tag{2}$$

holds for any $0 < r_1 < r_2 < r_0 := \sup\limits_{y \in f(D)} |y - y_0|$ and any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1\,. \tag{3}$$

Note that estimates of the type (2) are well known and hold at least for $p = n$ in many classes of mappings (see, e.g., [2, Theorem 3.2], [3, Theorem 6.7.II] and [4, Theorem 8.5]). For $p \neq n$, similar estimates may be found, e.g., in [5] and [6].

A mapping $f : D \to \mathbb{R}^n$ is called *discrete* if the image $\{f^{-1}(y)\}$ of any point $y \in \mathbb{R}^n$ consists of isolated points, and *open* if the image of any open set $U \subset D$ is an open set in $\mathbb{R}^n$.

Later, in the extended space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ we use the *spherical (chordal) metric* $h(x, y) = |\pi(x) - \pi(y)|$, where π is a stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ in $\mathbb{R}^{n+1}$, namely,

$$h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}\,,$$

$$h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}}\,, \quad x \neq \infty \neq y \tag{4}$$

(see, e.g., [1, Definition 12.1]). The following statement is true.

Theorem 1. *Let $n \geqslant 2$, $p \geqslant n$, let D be a domain in $\mathbb{R}^n$, $x_0 \in D$, and let $f : D \setminus \{x_0\} \to \mathbb{R}^n$ be an open discrete mapping that satisfies the conditions (2)-(3) at any point $y_0 \in \overline{D'} \setminus \{\infty\}$, where $D' := f(D \setminus \{x_0\})$.*

If $Q \in L^1(D')$, then f has a continuous extension $\overline{f} : D \to \overline{\mathbb{R}^n}$, the continuity of which should be understood in the sense of the chordal metric h in (4). The extended mapping $\overline{f}$ is open and discrete in D. Moreover, if $p = n$ and $\overline{f}(x_0) \neq \infty$, then there is a neighborhood $U \subset D$ of the point x_0 depending only on x_0, and $C = C(n, D, x_0) > 0$ such that

$$|\overline{f}(x) - \overline{f}(x_0)| \leqslant \frac{C_n \cdot (\|Q\|_1)^{1/n}}{\log^{1/n}\left(1 + \frac{\delta}{2|x - x_0|}\right)} \tag{5}$$

for any $x, y \in U$, where $\|Q\|_1$ is the norm of the function Q in $L^1(D')$.

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New Combinatorial Invariants of Doubly Periodic Tangles

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Doubly periodic tangles, or *DP tangles*, are complex entangled structures consisting of curves embedded in the thickened plane $\mathbb{E}^2 \times I$ that are periodically repeated in two directions. Hence, DP tangles can be defined as lifts of links in the thickened torus, $T^2 \times I$. The topological classification of DP tangles is at least as hard a problem as the full classification of knots and links in the three-space and is approached by constructing topological invariants. To reduce the complexity of this problem, the idea is to consider the quotient of a DP diagram under a periodic lattice, namely a link diagram in the (flat) torus T^2 that we call *(flat) motif.* This approach leads to a diagrammatic theory of the topological equivalence of DP tangles, which has been established in [1] on the level of motifs, and that generalizes works initiated by Grishanov et al. related to textiles ([4, 5]).

In this talk we will first establish the mathematical framework of the topological theory of DP tangles in order to characterize the notion of *equivalence* between DP tangles and between their flat motifs. Time permits, we shall further generalize these results to other diagrammatic categories, such as framed, virtual, singular, pseudo and bonded DP tangles, which could be used in novel applications. We will then introduce new topological invariants of DP tangles. In particular, we will present the notion of *axis-motif*, that is a set of arcs in the flat torus which can be viewed as a blueprint of a DP tangle capturing the different directions along which its components are organized. This will lead to the definition of the *directional type* of the DP tangle, which constitutes a topological invariant of DP tangles ([2]). We will also introduce the concept of *density* of a motif τ, defined in terms of the total number of arcs of the axis-motif of τ, which gives rise to a new invariant called *density of the DP tangle* τ_∞, defined as the minimal density over all axis-motifs of τ_∞. However, we will note that this topological invariant is not strong enough to distinguish two DP tangles of different directional types. Thus, by using the fact that the set of arcs of an axis-motif of a motif τ can be partitioned into a specific triple of integers, that we call *arc-triple of* τ, we will present a stronger invariant of DP tangles, called *minimal arc-triple.* This notion leads to a characterization of the directional type of a DP tangle by its minimal arc-triple. All the above invariants of DP tangles are measures that naturally inform on their topological complexity, they refer to global topological properties of theirs, and they add to the list of the existing invariants.

This is a joint work with Dr. Sonia Mahmoudi (Tohoku University, Japan) and Prof. Dr. Sofia Lambropoulou (National Technical University of Athens, Greece).

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Inequalities involving means in high-dimensional spaces

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The objects of study are convex bodies: compact, convex subsets of Euclidean spaces. Convexity naturally appears in many areas of mathematics, such as Linear Programming, Probability Theory, Functional Analysis, Partial Differential Equations, Information Theory and Geometry of Numbers.

For instance, density functions of some of the most important probability measures are logarithmically (or at least quasi) concave functions, like gaussians, exponential, or uniform densities over convex domains. In particular, this means that all their level sets are convex. Although convexity is a simple to formulate property, convex bodies possess a surprisingly rich structure. The main subject of the proposed project are geometric inequalities and extreme relations between convex sets in general. Especially, we are interested in extending results given so far only for symmetric convex sets or join results given separately for the symmetric case and the general case. To do so we want to take a functional into account that measures how far a convex body C is away from being symmetric. One such functional is the so called *Minkowski (measure of) asymmetry*, which measures in terms of the Banach-Mazur distance how far a set is from its closest symmetric set.

We start explaining some notation, which is mostly standard in convex geometry. For details see, e.g. [10]. For any $A, B \subset \mathbb{R}^n$ let $A \subset_t B$ denote that there exists a translate of A being a subset of B. The *Minkowski addition* of two sets $A, B \subset \mathbb{R}^n$ is given by the set $A + B = \{x + y : x \in A, y \in B\}$. Moreover, for any n-dimensional convex set K we denote by $\rho K := \{\rho x : x \in K\}$ and $-K := (-1)K$. For any set convex set K, we say that K is *symmetric* if $K =_t -K$. Moreover, let $K^\circ = \{x \in \mathbb{R}^n : x^\perp y \leq 1 \, \forall y \in K\}$ be the *polar* of K.

The main object of study in this project is the *Minkowski asymmetry* of a convex set C, defined as

$$s(C) = \min\{\rho > 0 : C \subset_t -\rho C\},$$

where we are allowed to write min instead of inf as C is a convex set, and this is true for all similar functionals we define below. Moreover, if $c - C \subset s(C)(c - C)$ we say that c is a *Minkowski center* of C, and if $c = 0$, we say that C is *Minkowski centered.* It is well known (see e.g. [8]) that for all convex sets C we have $s(C) \in [1, n]$ with $s(C) = 1$ if and only if C is symmetric and $s(C) = n$ if and only if C is an *n-simplex*, i.e., the convex hull of $n + 1$ affinely independent points.

Naturally, the Minkowski sum of two convex sets defines a mean. The harmonic, geometric, and arithmetic means of real numbers a and b are collectively known as the Pythagorean means. They are related by the extended arithmetic-geometric-harmonic mean inequality (see [10]). Thus, for convex sets K and C the *arithmetic mean* is defined as $\frac{K+C}{2}$, while the *harmonic mean* is defined as $\left(\frac{K^\circ+C^\circ}{2}\right)^\circ$. The *minimum* and *maximum* of K and C are represented by $K \cap C$ and $\operatorname{conv}(K \cup C)$, respectively. In the 1960s, Firey introduced and studied different means of convex sets, known as p-means (see [6, 7]). This line of investigation continues to this day (see [9]).

Notice that the considered symmetrizations of a convex body K, i.e., $K \cap (-K)$, $\frac{K-K}{2}$, $\operatorname{conv}(K \cup (-K))$, are frequently used in convex geometry, e.g., as extreme cases of a variety of geometric inequalities. Consider, e.g., the Bohnenblust inequality [1], which bounds from above the ratio of the circumradius ($\min_{x \in \mathbb{R}^n} \max_{y \in K} \|x-y\|$) and the diameter ($\max_{x,y \in K} \|x-y\|$) of convex bodies in arbitrary normed spaces endowed with a norm $\|\cdot\|$ by $n/(n+1)$, and for which equality is reached in spaces with $S \cap (-S)$ or $\frac{1}{2}(S-S)$ as the unit ball [5] where S is a 0-centered regular simplex. These means also appear in characterizations of spaces, for which K is complete or reduced [4, Prop. 3.5 – 3.10].

In [6] it was shown that similarly to the Pythagorean means, the means of convex sets can be ordered in terms of inclusions [6]. Thus, for any convex sets K, C with 0 in their interior we have

$$K \cap C \subset \left(\frac{K^\circ + C^\circ}{2}\right)^\circ \subset \frac{K+C}{2} \subset \operatorname{conv}(K \cup C). \tag{1}$$

For a Minkowski-centered convex compact set K we define the factor $\alpha(K)$ to be the smallest possible factor to cover $K \cap (-K)$ by $\operatorname{conv}(K \cup (-K))$, i.e.,

$$\alpha(K) := \inf\{\rho > 0 : K \cap (-K) \subset \rho \operatorname{conv}(K \cup (-K))\}.$$

In [2] we show a surprising result, showing that in 2-space the greatest value of the Minkowski asymmetry such that the harmonic mean can be optimally contained in the arithmetic mean is the golden ratio $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61$. Moreover, if $s(K) = \varphi$, there exists a non-singular linear transformation L, such that

$$L(K) = \operatorname{conv}\left(\left\{\begin{pmatrix}-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\\varphi\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}\right)$$

is the *golden house*.

We also present a family of planar sets K_s with $s(K_s) = s \in [1, \varphi]$, such that $\alpha(K) = 1$, thus, showing that for any $s \in [1,2]$ there exists a planar Minkowski centered K with $s(K) = s$, $\alpha(K) = 1$.

In [3] we give a complete description the region of all possible values for $\alpha(K)$ for planar Minkowski centered K in dependence of the asymmetry of K, showing that

$$\frac{2}{s(K)+1} \leq \alpha(K) \leq \min\left\{1, \frac{s(K)}{s(K)^2-1}\right\}.$$

Moreover, for every pair (α, s), such that $\frac{2}{s+1} \leq \alpha \leq \min\left\{1, \frac{s}{s^2-1}\right\}$, there exists a Minkowski centered planar convex set K, such that $s(K) = s$ and $\alpha(K) = \alpha$.

Surprisingly, in the same paper we were able to describe the number of intersection points of the boundaries of a convex set K and its negative $-K$, when its asymmetry is greater than the golden ratio. Namely, we show that for any Minkowski centered K with $s(K) \geq \varphi$ the set $\operatorname{bd}(K) \cap \operatorname{bd}(-K)$ consists of exactly 6 points. However, when the asymmetry is less than the golden ratio, $\operatorname{bd}(K) \cap \operatorname{bd}(-K)$ can consist of countable or uncountable number of points, as well as of a small one.

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Two problems in the theory of metric preserving functions

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The following is a particular case of J. Jachymski and F. Turoboś concept, see [1] for more details.

Definition 1. Let $\mathbf{A}$ be a class of metric spaces. Let us denote by $\mathbf{P_A}$ the set of all functions $f : [0, \infty) \to [0, \infty)$ such that the implication

$$((X, d) \in \mathbf{A}) \Rightarrow ((X, f \circ d) \in \mathbf{A})$$

is valid for every metric space (X, d).

We will use the following notations:

- $\mathbf{F}$, set of functions $f : [0, \infty) \to [0, \infty)$;
- $\mathbf{M}$, class of metric spaces;
- $\mathbf{U}$, class of ultrametric spaces;

Definition 2. A function $f \in \mathbf{F}$ is *metric preserving* (*ultrametric preserving*) iff $f \in \mathbf{P_M}$ ($f \in \mathbf{P_U}$).

Remark 3. The concept of metric preserving functions can be traced back to Wilson [2]. Similar problems were considered by Blumenthal in [3]. The theory of metric preserving functions was developed by Borsík, Doboš, Piotrowski, Vallin and other mathematicians. See also lectures by Doboš [4], and the introductory paper by Corazza [5]. The study of ultrametric preserving functions begun by P. Pongsriiam and I. Termwuttipong in 2014 [6].

Our main purpose is to give the answers on the following problems.

Problem 4. Let $\mathbf{A} \subseteq \mathbf{P_M}$. Find conditions under which the equation

$$\mathbf{P_X} = \mathbf{A} \tag{1}$$

has a solution $\mathbf{X} \subseteq \mathbf{M}$.

Problem 5. Let $\mathbf{A} \subseteq \mathbf{P_U}$. Find conditions under which equation (1) has a solution $\mathbf{X} \subseteq \mathbf{U}$.

Let us recall some basic concepts of semigroup theory, see, for example, John M. Howie [7].

A *semigroup* is a pair $(S, *)$ consisting of a nonempty set S and an associative operation $* : S \times S \to S$ which is called the *multiplication* on S. A semigroup $S = (S, *)$ is a *monoid* if there is $e \in S$ such that

$$e * s = s * e = s$$

for every $s \in S$.

Definition 6. Let $(S, *)$ be a semigroup and $\varnothing \neq T \subseteq S$. Then T is a *subsemigroup* of S if $a, b \in T \Rightarrow a * b \in T$. If $(S, *)$ is a monoid with the identity e, then T is a *submonoid* of S if T is a subsemigroup of S and $e \in T$.

Solutions to Problems 4 and 5 are given, respectively, in Theorems 7 and 8 below.

Theorem 7. *Let* $\mathbf{A}$ *be a nonempty subset of the set* $\mathbf{P_M}$ *of all metric preserving functions. Then the following statements are equivalent.*

(*i*) *The equality*

$$\mathbf{P_X} = \mathbf{A} \tag{2}$$

has a solution $\mathbf{X} \subseteq \mathbf{M}$.

(*ii*) $\mathbf{A}$ *is a submonoid of* $(\mathbf{F}, \circ)$.

(*iii*) $\mathbf{A}$ *is a submonoid of* $(\mathbf{P_M}, \circ)$.

The next theorem is an ultrametric analog of the previous theorem.

Theorem 8. *Let* $\mathbf{A}$ *be a nonempty subset of the set* $\mathbf{P_U}$ *of all ultrametric preserving functions. Then the following statements are equivalent.*

(*i*) *The equality* $\mathbf{P_X} = \mathbf{A}$ *has a solution* $\mathbf{X} \subseteq \mathbf{U}$.

(*ii*) $\mathbf{A}$ *is a submonoid of* $(\mathbf{F}, \circ)$.

(*iii*) $\mathbf{A}$ *is a submonoid of* $(\mathbf{P_U}, \circ)$.

Some properties of the monoids of $\mathbf{P_M}$ and $\mathbf{P_U}$ were described in [8] and [9].

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Group action on noncommutative curves

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Recall that a *noncommutative curve* (*noc*) is a pair $\mathbb{X}=(X,\mathcal{O}_{\mathbb{X}})$, where X is an algebraic curve (the *base curve* of $\mathbb{X}$) over a filed $\Bbbk$ and $\mathcal{O}_{\mathbb{X}}$ is a sheaf of $\mathcal{O}_X$-algebras coherent as a sheaf of $\mathcal{O}_X$-modules. We always suppose that $\mathcal{O}_X\subseteq\mathcal{O}_{\mathbb{X}}$ and the curve $\mathbb{X}$ is *reduced*, i.e. $\mathcal{O}_{\mathbb{X}}$ has no nilpotent ideals. We also suppose that $\Bbbk$ is algebraically closed.

The noc $\mathbb{X}$ is called *hereditary* if $\mathrm{gl.dim}\,\mathcal{O}_{\mathbb{X}}=1$, that is $\mathcal{E}xt^n_{\mathcal{O}_{\mathbb{X}}}(\mathcal{M},\mathcal{N})=0$ for $n>1$ and all coherent sheaves of $\mathcal{O}_{\mathbb{X}}$-modules $\mathcal{M},\mathcal{N}$. We denote by $\mathbb{X}^\sharp$ the noc $(\mathcal{O}_X,\mathcal{O}_{\mathbb{X}^\sharp})$, where for every point $x\in X$

$$\mathcal{O}_{\mathbb{X}^\sharp,x}=\begin{cases}\mathcal{O}_{\mathbb{X},x} & \text{if } \mathcal{O}_{\mathbb{X},x}\text{is hereditary,}\\ \mathcal{E}nd_{\mathbb{X},x}\,\mathfrak{r}_x & \text{otherwsise,}\end{cases}$$

where $\mathfrak{r}_x=\mathrm{rad}\,\mathcal{O}_{\mathbb{X},x}$. It is known that if $\mathfrak{r}_x=\mathrm{rad}\,\mathcal{O}_{\mathbb{X}^\sharp,x}$, the noc $\mathbb{X}$ is hereditary. In this case $\mathbb{X}$ is called a *Bäckström curve*. If, moreover, $\ell_{\mathcal{O}_{\mathbb{X}}}\big(\mathcal{O}_{\mathbb{X}^\sharp}\otimes_{\mathcal{O}_{\mathbb{X}}}U\le 2$ for every simple sheaf of $\mathcal{O}_{\mathbb{X}}$-modules U, $\mathbb{X}$ is called a *nodal curve*. Note that a "usual" (commutative) curve X is nodal if and only if all its singularities are *simple nodes*, that is, if $x\in X$ is a singular point, $\hat{\mathcal{O}}_x\simeq\Bbbk[[x,y]]/(xy)$. The structure of nodal nocs is described in [1].

Let a finite group G acts on a noc $\mathbb{X}$. It means that it acts on the base curve X as well as on the sheaf of algebras $\mathcal{O}_{\mathbb{X}}$ (maybe with a factor set in the sense of [2]). The the noc $\mathbb{X}*G=(X/G,\mathcal{O}_{\mathbb{X}*G})$ (the *crossed product* of $\mathbb{X}$ and G) is defined. Note that G naturally acts on the category $\mathrm{Coh}\,\mathbb{X}$ of coherent $\mathcal{O}_{\mathbb{X}}$-modules.

Theorem 1. *Suppose that the order of the group G is invertible in $\Bbbk$.*

- *(1) There is an equivalence of categories* $\mathrm{Coh}(\mathbb{X}*G)$ *and* $\mathrm{add}((\mathrm{Coh}\,\mathbb{X})*G)$, *where* $\mathrm{add}\,\mathcal{C}$ *denotes the* Karubian closure *of the category $\mathcal{C}$, i.e. the smallest additive category containing $\mathcal{C}$ and such that all idempotents in it split.*
- *(2) If $\mathbb{X}$ is hereditary (Bäckström, nodal), so is $\mathbb{X}*G$.*

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$6D$-Riemannian metric associated at the Navier-Stokes equations and its applications

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Theorem 1. *The $6D$ metric in local coordinates (t,x,y,z,v,w)*

$${d_s}^2=-2\,B\left(t,x,y,z\right)d_td_v+2\,E\left(t,x,y,z\right)d_td_w+d_td_x+2\,H\left(t,x,y,z\right)d_vd_w+d_vd_y-$$

$$-2\int\frac{\partial}{\partial y}H\left(t,x,y,z\right)\,\mathrm{d}z{d_w}^2+d_wd_z,$$

with conditions on coefficients

$$\frac{\partial}{\partial y}H\left(t,x,y,z\right)-\frac{\partial}{\partial x}E\left(t,x,y,z\right)=0,\ \frac{\partial}{\partial z}H\left(t,x,y,z\right)-\frac{\partial}{\partial x}B\left(t,x,y,z\right)=0,$$
$$\frac{\partial}{\partial z}E\left(t,x,y,z,t\right)-\frac{\partial}{\partial y}B\left(t,x,y,z\right)=0$$

is associated with equations of compatibility of the Navier-Stokes system o equations

$$\vec{\nabla}P(\vec{t},x)=\frac{\partial}{\partial t}\vec{V}+\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}=\mu\Delta\vec{V},\qquad\qquad \vec{\nabla}\cdot\vec{V}=0$$

with respect to the function of pressure in flow of liquid $P(\vec{t},x)=P(t,x,y,z)$. [1],[2]

On base of this theorem the examples of exact solutions of Navier-Stokes system of equations may be constructed.

The Ricci-tensor R_{ik} of given metric has six components $R_{tt},R_{vv},R_{vw},R_{ww},R_{tv},R_{tw}$ and from conditions of compatibility between the various equations for R_{ik} lead determined the components of velocities $\vec{U}(t,\vec{x})$ and the function pressure $P(t,\vec{x})$.

As example simplest reduction of considered metric has the form

$$ds^2=-2\left(\frac{\partial^2}{\partial z^2}K\left(t,x,y,z\right)\right)d_td_v+2\left(\frac{\partial^2}{\partial z\partial y}K\left(t,x,y,z\right)\right)d_td_w+d_td_x+$$
$$+2\left(\frac{\partial^2}{\partial z\partial x}K\left(t,x,y,z\right)\right)d_vd_w+d_vd_y-2\left(\frac{\partial^2}{\partial y\partial x}K\left(t,x,y,z\right)\right){d_w}^2+d_wd_z.$$

As example the metric is the Ricci-flat for the flows of the form

$$U\left(t,x,y,z\right)=-1/2\,x\frac{\partial}{\partial z}F\left(t,z\right),\quad V\left(t,x,y,z\right)=-1/2\,y\frac{\partial}{\partial z}F\left(t,z\right),\quad W\left(t,x,y,z\right)=F\left(t,z\right),$$

with the function $F\left(t,z\right)$ which is determined from the equation

$$-\mu\frac{\partial^3}{\partial z^3}+F\left(t,z\right)\frac{\partial^3}{\partial z^3}F\left(t,z\right)+\frac{\partial^2}{\partial z\partial t}F\left(t,z\right)=0,$$

for which

$$B(t,\vec{x})=\frac{\partial}{\partial t}F\left(t,z\right)+F\left(t,z\right)\frac{\partial}{\partial z}F\left(t,z\right)$$
$$H(t,\vec{x})=-1/2x\frac{\partial^2}{\partial z\partial t}F\left(t,z\right)+1/4x\frac{\partial}{\partial z}F\left(t,z\right)^2-$$
$$-1/2xF\left(t,z\right)\frac{\partial^2}{\partial z\partial z}F\left(t,z\right)+1/2\,\mu x\frac{\partial^3}{\partial z\partial z\partial z}F\left(t,z\right),$$
$$E(t,\vec{x})=-1/2y\frac{\partial^2}{\partial z\partial t}F\left(t,z\right)+1/4y\frac{\partial}{\partial z}F\left(t,z\right)^2-$$
$$-1/2yF\left(t,z\right)\frac{\partial^2}{\partial z\partial z}F\left(t,z\right)+1/2\,\mu y\frac{\partial^3}{\partial z\partial z\partial z}F\left(t,z\right).$$

The metric of the form

Theorem 2.

$$ds^2 = 2\,dx^2 + 2\,dxdy + 2\,dxdu + 2\,dy^2 + 2\,dydz + 2\,dydv + 2\,dz^2 + 2\,dzdw + \\ + 2\,dtdp + 2\,d\eta d\xi + 2\,d\rho d\chi + 2\,dmdn + A dt^2 + B d\eta^2 + C d\rho^2 + E dm^2, \tag{1}$$

where

$$A = 2 - U\left(x, y, z, t\right) u - V\left(x, y, z, t\right) v - W\left(x, y, z, t\right) w,$$

$$B = \left(-UW + \mu\,\frac{\partial}{\partial z}U\right) w + \left(-UV + \mu\,\frac{\partial}{\partial y}U\right) v + \left(\mu\,\frac{\partial}{\partial x}U - (U)^2 - P\right) u - Up,$$

$$C = \left(-VW + \mu\,\frac{\partial}{\partial z}V\right) w + \left(\mu\,\frac{\partial}{\partial y}V - (V)^2 - P\right) v + \left(-UV + \mu\,\frac{\partial}{\partial x}V\right) u - Vp,$$

$$E = \left(-\mu\,\frac{\partial}{\partial x}U - \mu\,\frac{\partial}{\partial y}V - (W)^2 - P\right) w + \left(-VW + \mu\,\frac{\partial}{\partial y}W\right) v + \left(-UW + \mu\,\frac{\partial}{\partial x}W\right) u - Wp$$

is the Ricci-flat on solutions of Navier-Stokes system of equations.

Geometric characteristics of the 6D- metric depend on choice of the functions $H(t, \vec{x}), B(t, \vec{x}), E(t, \vec{x})$ and considerations of their properties joint with the metric of 14D-space can be used to determination of the functions

$$U(t, \vec{x}), \qquad V(t, \vec{x}), \qquad W(t, \vec{x}), \qquad P(t, \vec{x})$$

from the NS-equations:

$$\frac{\partial}{\partial t}U\left(t, \vec{x}\right) + U\left(t, \vec{x}\right)\frac{\partial}{\partial x}U\left(t, \vec{x}\right) + V\left(t, \vec{x}\right)\frac{\partial}{\partial y}U\left(t, \vec{x}\right) + W\left(t, \vec{x}\right)\frac{\partial}{\partial z}U\left(t, \vec{x}\right) - \\ - \mu\left(\frac{\partial^2}{\partial x^2}U\left(t, \vec{x}\right) + \frac{\partial^2}{\partial y^2}U\left(t, \vec{x}\right) + \frac{\partial^2}{\partial z^2}U\left(t, \vec{x}\right)\right) + H\left(t, \vec{x}\right) = 0,$$

$$\frac{\partial}{\partial t}V\left(t, \vec{x}\right) + U\left(t, \vec{x}\right)\frac{\partial}{\partial x}V\left(t, \vec{x}\right) + V\left(t, \vec{x}\right)\frac{\partial}{\partial y}V\left(t, \vec{x}\right) + W\left(t, \vec{x}\right)\frac{\partial}{\partial z}V\left(t, \vec{x}\right) - \\ - \mu\left(\frac{\partial^2}{\partial x^2}V\left(t, \vec{x}\right) + \frac{\partial^2}{\partial y^2}V\left(t, \vec{x}\right) + \frac{\partial^2}{\partial z^2}V\left(t, \vec{x}\right)\right) + E\left(t, \vec{x}\right) = 0,$$

$$\frac{\partial}{\partial t}W\left(t, \vec{x}\right) + U\left(t, \vec{x}\right))\frac{\partial}{\partial x}W\left(t, \vec{x}\right) + V\left(t, \vec{x}\right)\frac{\partial}{\partial y}W\left(t, \vec{x}\right) + W\left(t, \vec{x}\right)\frac{\partial}{\partial z}W\left(t, \vec{x}\right) - \\ - \mu\left(\frac{\partial^2}{\partial x^2}W\left(t, \vec{x}\right) + \frac{\partial^2}{\partial y^2}W\left(t, \vec{x}\right) + \frac{\partial^2}{\partial z^2}W\left(t, \vec{x}\right)\right) + B\left(t, \vec{x}\right) = 0$$

$$\frac{\partial}{\partial x}U\left(t, \vec{x}\right) + \frac{\partial}{\partial y}V\left(t, vecx\right) + \frac{\partial}{\partial z}W\left(t, x, y, z\right) = 0.$$

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Interplay of Global Implicit Functions and Critical Point Theory in Infinite Dimensional Spaces

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Consider a nonlinear equation of the form:

$$\Phi(\mathsf{e}, \mathsf{g}) = \mathbf{0}, \tag{1}$$

where e, g, and $\mathbf{0}$ belong to arbitrary Fréchet spaces, and $\mathbf{0}$ represents the zero element. We establish sufficient conditions under which it is possible to globally and uniquely solve Equation (1) for g in terms of e, with the solution mapping $\mathcal{K}$ being differentiable, such that Φ does not lose the derivative.

Applying the obtained global implicit function theorem, we will establish sufficient conditions for the global existence and uniqueness of the solution over the entire time of the following initial value problem that involves the loss of one derivative:

$$y'(t) = \Phi(t, y(t), e), \tag{2}$$

where the initial conditions are fixed both in time and in arbitrary Fréchet spaces.

We also generalize the Lagrange multiplier method, which involves finding critical points of a mapping subject to a set of constraints, and apply the results to extend the Nehari method for locating critical points.

The full details can be found in [1].

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Spherical Analysis on Fuzzy Lie Groups

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Let G be a locally compact Lie group and $\mathfrak{g}$ its Lie algebra. We consider a fuzzy analogue of G, denoted by $\mathfrak{G}_\mathfrak{f}$ called a fuzzy Lie group. Spherical functions on $\mathfrak{G}_\mathfrak{f}$ are constructed and a version of the existence result of the Helgason-spherical function on G is then established on $\mathfrak{G}_\mathfrak{f}$.

On the construction and classification of the common invariant solutions for the $(1+3)$-dimensional Euler-Lagrange-Born-Infeld and homogeneous Monge-Ampere equations

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A solution of many problems of the theory of minimal surfaces, nonlinear electrodynamics, geometric optics, theories of gravity, geometry, unified field theory, string theories, black holes, cosmology, etc. is reduced to the investigation of the Euler-Lagrange equations [1, 2, 3, 4, 5, 6, 7], the Born-Infeld equations [8, 9, 10, 11, 12, 13], the Monge-Ampère equations [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27] in the spaces of different dimensions and different types.

We consider the following $(1+3)$-dimensional PDEs:

- the Euler-Lagrange-Born-Infeld equation

$$\Box u\,(1-u_\nu u^\nu)+u^\mu u^\nu u_{\mu\nu}=0,$$

- the homogeneous Monge-Ampère equation

$$\det\left(u_{\mu\nu}\right)=0,$$

where $u=u(x)$, $x=(x_0,x_1,x_2,x_3)\in M(1,3)$, $u_\mu\equiv\dfrac{\partial u}{\partial x^\mu}$, $u_{\mu\nu}\equiv\dfrac{\partial^2 u}{\partial x^\mu\partial x^\nu}$, $u^\mu=g^{\mu\nu}u_\nu$, $g_{\mu\nu}=(1,-1,-1,-1)\delta_{\mu\nu}$, $\mu,\nu=0,1,2,3$, and $\Box$ is the d'Alembert operator.

Here, $M(1,3)$ is a four-dimensional Minkowski space.

From the results obtained by W.I. Fushchich, W.M. Shtelen and N.I. Serov [28] it follows, in particular, that the common symmetry group of those equations is the generalized Poincaré group $P(1,4)$.

At the present time, we have constructed as well as classified some common invariant solutions of the equations under study. To obtain those results, we have used the results of the classification of symmetry reductions and invariant solutions of the Euler–Lagrange–Born–Infeld equation [29, 30].

In our report, I plan to present some of the results obtained.

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Normal forms of functions with degenerate critical points on surfaces whose stabilizers are homotopically non-trivial

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Let M be a smooth compact surface, and P be either $\mathbb{R}$ or S^1. The group of diffeomorphisms $\mathcal{D}(M)$ acts on the space of smooth P-valued functions $C^\infty(M,P)$ by the rule:

$$C^\infty(M,P)\times\mathcal{D}(M)\to C^\infty(M,P)\qquad (f,h)\mapsto f\circ h.$$

For a smooth function $f\in C^\infty(M,P)$ we denote by

$$\mathcal{S}(f)=\{h\in\mathcal{D}(M)\,|\,f\circ h=f\},\qquad \mathcal{O}(f)=\{f\circ h\,|\,h\in\mathcal{D}(M)\}$$

the *stabilizer* and the *orbit* of f. Homotopy properties of $\mathcal{S}(f)$ and $\mathcal{O}(f)$ and their connected components are well studied for a large class of smooth functions with isolated singularities on surfaces, see [2]. We also denote by $\mathcal{S}_{\mathrm{id}}(f)$ a connected component of the identity map id in $\mathcal{S}(f)$.

We consider the following class $\mathcal{F}(M,P)$ of smooth functions: a function f belongs to $\mathcal{F}(M,P)$ if

(1) for each connected component V of the boundary ∂M a function $f|_V$ either takes a constant value or is a covering map,
(2) a set of critical points Σ_f of f is a disjoint union of smooth submanifolds of M and $\Sigma_f\subset\mathrm{Int}(M)$,
(3) for each connected component C of Σ_f and each critical point $p\in C$ there exist a local chart $(U,\phi:U\to\mathbb{R}^2)$ near p and a chart $(V,\psi:V\to\mathbb{R})$ near $f(p)\in P$ such that $f(U)\subset V$ and a local representation $f_p=\psi\circ f\circ\phi^{-1}:\phi(U)\to\psi(V)$ of f is
    (a) either a polynomial homogeneous polynomial f_p without multiple factors,
    (b) or is given by $f_C(x,y)=\pm y^{n_C}$ for some $n_C\in\mathbb{N}_{\geq 2}$ depending of C.

Connected components of Σ_f are isolated critical points and critical circles.

Let $\mathcal{F}^0(M,P)$ be a subset of $\mathcal{F}(M,P)$ of function which satisfy (1), (2), (3.b), but instead (3.a) the following condition holds:

(3.a') either a polynomial f_p given by $f_p(x,y)=\pm x^2\pm y^2$.

For a function $f\in\mathcal{F}(M,P)$ a stabilizer $\mathcal{S}_{\mathrm{id}}(f)$ is homotopy equivalent to S^1 if $f\in\mathcal{F}^0(M,P)$, and is contractible otherwise, [1, Theorem 1.2]. Our main result is an analytical characterization of functions from $\mathcal{F}^0(M,P)$, see Theorem 3. The following proposition contains basic facts about functions from $\mathcal{F}^0(M,P)$.

Proposition 1. *Let f be a function from $\mathcal{F}^0(M,P)$. Then*

(1) *M is one of the following surfaces: $S^1\times[0,1]$, D^2, S^2 or T^2.*
(2) *a function $f:M\to P$ is always null-homotopic if M is not a torus. A function on torus can be either null-homotopic or not null-homotopic.*
(3) *f has any finite number of critical circles if $M=S^1\times[0,1]$, D^2, S^1, or T^2 and f is not null-homotopic. If $M=T^2$ and f is null-homotopic, then f has at least 2 critical circles.*
(4) *If $M=S^1\times[0,1]$ or T^2, then f does not have isolated critical points. If $M=D^2$ or S^2, a function f has one and two non-degenerate extremes respectively.*

To state our main result we need the following definition.

Definition 2 (Primitive functions)**.** Let $f_0:M\to P$ be a smooth function

(1) $M = S^1 \times [0,1] = \{(z,s) \,|\, z \in \mathbb{C}, |z| = 1,\, 0 \leq s \leq 1\}$, and $f_0 : S^1 \times [0,1] \to \mathbb{R}$ is given by $f_0(\phi, s) = s$;
(2) $M = D^2 = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1\}$, and $f_0 : D^2 \to \mathbb{R}$ is given by $f_0(x,y) = \pm x^2 \pm y^2$;
(3) $M = S^2 = \{(x,y,z) \,|\, x^2 + y^2 + z^2 = 1\}$, and $f_0(x,y,z) : S^2 \to \mathbb{R}$ is given by $f_0(x,y,z) = z$;
(4) $M = T^2 = \left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, \left(\sqrt{x^2+y^2} - 2\right)^2 + z^2 = 1\right\}$, and $f_0 : T^2 \to \mathbb{R}$ is given by $f_0(x,y,z) = z$;
(5) $M = T^2 = \{(w,z) \in \mathbb{C}^2 \,|\, |z| = |w| = 1\}$, and $f_0 : T^2 \to S^1$ is given by $f_0(w,z) = z$.

Obviously that functions from (1)–(4) belongs to $\mathcal{F}^0(M,P)$. They are height functions for (1)–(4), and a function from (5) is an angular projection. These functions have a minimum possible number of critical submanifolds, and we will call them *primitive functions*.

Theorem 3. *Let f be a smooth function from $\mathcal{F}^0(M,P)$ and $f_0 \in \mathcal{F}^0(M,P)$ be a primitive function. A function f admits a decomposition*

$$f = \varkappa \circ f_0 \circ h^{-1} \tag{1}$$

for some diffeomorphism $h : M \to M$ and a smooth function $\varkappa : \mathrm{Im}(f_0) \to P$ satisfying the following conditions:

(A) *$\varkappa$ has the only finite number of critical points in which it is not flat, i.e., not all derivatives of $\varkappa$ at each critical point vanish,*
(B) *$\varkappa$ does not have extremes at $f_0(\Sigma^0_{f_0})$ and $f_0(\partial M)$.*

In particular, if $f \in \mathcal{F}^0(T^2,P)$ is null-homotopic, then f_0 is given by (4), and by (5) otherwise. A factorization (1) is not unique and depends on the choice of h.

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Nonlinear interpolation of α-Holderian mappings with applications to quasilinear PDEs

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The Marcinkiewicz interpolation theorems for linear operators acting on Lebesgue spaces turned out to be a powerful tool for studying regularity of solutions for linear PDEs in L^p-spaces. The K-method introduced by J. Peetre ([5, 6]) allowed to extend the study of regularity of solutions of linear equations on spaces different from L^p-spaces. The main difficulty to apply Peetre's definition is the identification of the interpolation spaces between two normed spaces embedded in a same topological space. In [2, 3, 4] we did such a study with applications to linear PDEs using new non-standard spaces as grand or small Lebesgue spaces and $G\Gamma$-gamma spaces.

In [7] L. Tartar gave interpolation results on nonlinear Hölderian mappings (which include Lipschitz mappings) and applied them to a variety of boundary value problems as bilinear applications, semi-linear PDEs but also on variational inequalities.

In this talk we present some results contained in the recent paper [1], were we extend Tartar's results on nonlinear interpolation of α-Hölderian mappings $\mathcal{T}$ by studying the action of the mappings $\mathcal{T}$ on K-functionals and between interpolation spaces with logarithm functors. Therefore, we identify some interpolation spaces using couples of Lebesgue or Lorentz spaces, recovering spaces as Lorentz–Zygmund spaces or $G\Gamma$-gamma spaces.

We apply these results to obtain regularity on the gradient of the weak or entropic-renormalized solution u to quasilinear equations of the form

$$-\text{div}(\widehat{a}(\nabla u)) + V(x; u) = f, \qquad u = 0 \text{ on } \partial\Omega, \tag{1}$$

associated to the Dirichlet homogeneous condition on the boundary, where Ω is a bounded smooth domain of $\mathbb{R}^n$, $\widehat{a}(\nabla u) = |\nabla u|^{p-2}\nabla u$, V is a nonlinear potential and f belongs to non-standard spaces like Lorentz-Zygmund spaces. We also show that the mapping $\mathcal{T}: \ \mathcal{T}f = \nabla u$ is locally or globally α-Hölderian under suitable values of α and appropriate assumptions on V and $\widehat{a}$.

Furthermore, also the anisotropic version or the variable exponents version of the Laplacian are considered.

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Riemann Integration on a space with a fractal structure

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In this work we start developing a Riemann-type integration theory on spaces which are equipped with a fractal structure (see [1] for more details). The definition of a fractal structure is the next one:

Definition 1. A fractal structure $\mathbf{\Gamma}$ on a set X is a countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ such that Γ_{n+1} is a strong refinement of Γ_n for each $n \in \mathbb{N}$. Γ_2 is said to be a strong refinement of Γ_1 if Γ_2 is a refinement of Γ_1 (that is, each element of Γ_2 is contained in some element of Γ_1) and for each $B \in \Gamma_1$ it holds that $B = \bigcup\{A \in \Gamma_2 : A \subseteq B\}$. Cover Γ_n is called level n of the fractal structure.

We require to define a concept first:

Definition 2. Let $(X, \mathcal{S}, \mu)$ be a measure space and $\mathbf{\Gamma}$ be a fractal structure on X. $\mathbf{\Gamma}$ is said to be μ-disjoint if the following conditions hold:

(1) $\Gamma_n \subseteq \mathcal{S}$ is countable for each $n \in \mathbb{N}$.
(2) $\mu(B \cap J) = 0$ for each $B, J \in \Gamma_n$ such that $B \neq J$ and each $n \in \mathbb{N}$.
(3) $\mu(A) < \infty$ for each $A \in \Gamma_n$ and each $n \in \mathbb{N}$.

Next, we define the Darboux sums with respect to a measure and a fractal structure:

Definition 3. Let $(X, \mathcal{S}, \mu)$ be a measure space, $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be a μ-disjoint fractal structure, and $f : X \to \mathbb{R}$ be a bounded function. Then, for each $J \in \Gamma_n$, we set $m(f; J) = \inf\{f(x) : x \in J\}$ and $M(f; J) = \sup\{f(x) : x \in J\}$, so that the lower and upper Darboux sums with respect to μ for each level of the fractal structure are given by

$$L(f; \Gamma_n, \mu) = \sum_{J \in \Gamma_n} m(f; J)\mu(J) \qquad \text{and} \qquad U(f; \Gamma_n, \mu) = \sum_{J \in \Gamma_n} M(f; J)\mu(J).$$

The lower and upper Riemann integrals with respect to a measure and a fractal structure are defined as follows:

Definition 4. Let $(X, \mathcal{S}, \mu)$ be a measure space, $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be a μ-disjoint fractal structure on X, and $f : X \to \mathbb{R}$ be a bounded function. We define the lower and upper Riemann integrals of f with respect to μ and $\mathbf{\Gamma}$ on X as follows:

(1) Upper Riemann integral of f with respect to μ and $\mathbf{\Gamma}$:

$$\overline{\int_X}^{(\mu,\mathbf{\Gamma})} f := \inf\{U(f; \Gamma_n; \mu) : n \in \mathbb{N}\} = \lim_n U(f; \Gamma_n; \mu).$$

(2) Lower Riemann integral of f with respect to μ and $\mathbf{\Gamma}$:

$$\underline{\int_X}^{(\mu,\mathbf{\Gamma})} f := \sup\{L(f; \Gamma_n; \mu) : n \in \mathbb{N}\} = \lim_n L(f; \Gamma_n; \mu).$$

Now we give the defition of a Riemann-integrable function.

Definition 5. Let $(X, \mathcal{S}, \mu)$ be a measure space, $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be a μ-disjoint fractal structure on X and $f : X \to \mathbb{R}$ be a bounded function. f is said to be Riemann-integrable with respect to μ and $\mathbf{\Gamma}$ on X if $\overline{\int_X}^{(\mu,\mathbf{\Gamma})} f$ is finite and $\underline{\int_X}^{(\mu,\mathbf{\Gamma})} f = \overline{\int_X}^{(\mu,\mathbf{\Gamma})} f$.

If f is Riemann-integrable with respect to μ and $\mathbf{\Gamma}$ on X, we define the Riemann integral of f with respect to μ and $\mathbf{\Gamma}$ on X, $\int_X^{(\mu,\mathbf{\Gamma})} f$, by $\int_X^{(\mu,\mathbf{\Gamma})} f = \underline{\int_X}^{(\mu,\mathbf{\Gamma})} f = \overline{\int_X}^{(\mu,\mathbf{\Gamma})} f$. We denote by $R(X; \mu; \mathbf{\Gamma})$ the set of Riemann-integrable functions with respect to μ and $\mathbf{\Gamma}$ on X.

The next step is defining the Riemann sum relative to a collection of points in a certain level of $\mathbf{\Gamma}$.

Definition 6. Let $\mathbf{\Gamma}$ be a fractal structure on a space X such that Γ_n is countable for each $n \in \mathbb{N}$. A selection for Γ_n is a collection of points $\xi := (x_A)_{A \in \Gamma_n}$ such that $x_A \in A$ for each $A \in \Gamma_n$.

Definition 7. Let $(X, \mathcal{S}, \mu)$ be a measure space, $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be a μ-disjoint fractal structure on X and $f : X \to \mathbb{R}$ be a bounded function. Let $n \in \mathbb{N}$ and $\xi = (x_A)_{A \in \Gamma_n}$ be a selection for Γ_n. The Riemann sum for f relative to Γ_n, ξ and μ is defined as $S(f; \Gamma_n; \xi; \mu) := \sum_{A \in \Gamma_n} f(x_A)\mu(A)$.

The following theorem is analogous to the Riemann's Theorem in $\mathbb{R}^n$, but for bounded functions defined on a space with a μ-disjoint fractal structure.

Theorem 8. *Let* $(X, \mathcal{S}, \mu)$ *be a measure space,* $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ *be a* μ*-disjoint fractal structure on* X*,* $f : X \to \mathbb{R}$ *be a bounded function and* $C \in \mathbb{R}$*. The following statements are equivalent:*

(1) $f \in R(X; \mu; \mathbf{\Gamma})$ *and* $\int_X^{(\mu,\mathbf{\Gamma})} f = C$.
(2) Given $\varepsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that* $|C - S(f; \Gamma_n; \xi_n; \mu)| < \varepsilon$ *for each* $n \geq n_0$ *and each selection for* Γ_n*,* ξ_n*.*
(3) Given $\varepsilon > 0$*, there exists* $n \in \mathbb{N}$ *such that* $|C - S(f; \Gamma_n; \xi; \mu)| < \varepsilon$ *for each selection for* Γ_n*,* ξ*.*
(4) $S(f; \Gamma_m; \xi_m; \mu) \overset{m\to\infty}{\longrightarrow} C$ *for each sequence* (ξ_m) *such that* ξ_m *is a selection for* Γ_m *for each* $m \in \mathbb{N}$*.*

The next result is crucial in order to justify that the Riemann integral of a bounded function with respect to a measure and a fractal structure does not depend on the fractal structure.

Proposition 9. *Let* $(X, \mathcal{S}, \mu)$ *be a measure space,* $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ *and* $\mathbf{\Gamma}^* = \{\Gamma_n^* : n \in \mathbb{N}\}$ *be two* μ*-disjoint fractal structures on* X *and* $f : X \to \mathbb{R}$ *be a bounded function. If* $f \in R(X; \mu; \mathbf{\Gamma})$ *and* $f \in R(X; \mu; \mathbf{\Gamma}^*)$*, then* $\int_X^{(\mu,\mathbf{\Gamma})} f = \int_X^{(\mu,\mathbf{\Gamma}^*)} f$*.*

Hence, it does make sense to introduce the following concept:

Definition 10. Let $(X, \mathcal{S}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ be a bounded function. f is said to be μ-Riemann-integrable if there exists a μ-disjoint fractal structure $\mathbf{\Gamma}$ on X such that f is Riemann-integrable on X with respect to μ and $\mathbf{\Gamma}$. Moreover, if so, the integral is defined as $\int_X^{\mu} f = \int_X^{(\mu,\mathbf{\Gamma})} f$.

Proposition 11. *Let* $(X, \mathcal{S}, \mu)$ *be a finite measure space and* $f : X \to \mathbb{R}$ *be a bounded measurable function. Then* $f \in R(X; \mu)$ *and* $\int_X^{\mu} f = \int f d\mu$*.*

Hence, if $\mathbf{\Gamma}$ is a μ-disjoint fractal structure on X such that f is Riemann-integrable with respect to μ and $\mathbf{\Gamma}$, we can calculate $\int f d\mu$ as $\int_X^{(\mu,\mathbf{\Gamma})} f$. It also follows that if μ is a finite measure on the Borel σ-algebra of a topological space X and $f : X \to \mathbb{R}$ is a bounded continuous map, then f is μ-Riemann-integrable on X.

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Topological rigidity of quoric manifolds

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The basic problem in Geometric Topology is the topological classification of manifolds, spaces that are locally like the usual Euclidean spaces, like the surfaces. More precisely, we study manifolds that have the same algebraic properties (homotopy equivalences) and we would like to show that they are equivalent (homeomorphic). There are a lot of conjectures towards this direction with the strongest being the Isomorphism Conjecture of Farrell-Jones. Furthermore, there are the corresponding conjectures when the manifolds are equipped with a group of symmetries (group actions). In this case, all the structures (homotopy equivalences, homeomorphisms) should preserve the group action (equivariant).

The original idea of the classification problems is Mostow's Rigidity Theorem in which it was proved that two hyperbolic manifolds, of dimension larger than 2, which are homotopy equivalent, they are isometric. This result is the basis of most of the conjectures of classification and rigidity. Usually, one of the two manifolds has nice properties (nonpositive curvature, hyperbolic fundamental group) and the other is simply homotopy equivalent to the first. The problem is to equip the second manifold with the properties of the first through the homotopy equivalence. After that, geometric methods, similar to the one in Mostow's Theorem, will give the result.

In the case of interest, we start with Euclidean spaces $\mathbb{R}^n$ on which we can define a multiplication such that, if $x, y \neq 0$, then $xy \neq 0$. If we insist that the multiplication is associative, then $n = 1, 2, 4$ from Frobenious Theorem. In the first case we have the multiplication of the reals, in the second case we have the complex multiplication and in the third case, we have the multiplication on the quaternions which is not commutative. The corresponding spheres, in each case, are the elements of length one and for $n = 1$ is the group of two elements $\mathbb{Z}_2$, for $n = 2$ is the unit circle S^1, for $n = 4$ is the unit sphere in $\mathbb{R}^4$, S^3, and they are all groups with the induced multiplication. In each of these dimension we have the corresponding torus, $Z^n = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, $T^n = S^1 \times \ldots \times S^1$ and $Q^n = S^3 \times \ldots \times S^3$. In each case, starting with a polyhedron, we can construct a space on which the tori act and the quotient space in the original polyhedron. These manifolds are called standard models. The local action is given in fact as the corresponding multiplication.

In each case, we start with a manifold on which the tori act (locally linearly) and they are homotopy equivalent to the standard model, preserving the group action and we want to show that is homeomorphic to the standard model. In all case, the process is similar. Let N be the manifold that we study. First, we show that the action has the local properties of the standard model. Thus the quotient space is a polyhedron P. Next, we construct the standard model from P. The final result is consequence of two results: first that N is homeomorphic to the canonical model over P, which is homeomorphic to the original standard model.

For $n = 1$, we have a much richer structure than those of finite groups. In this case, we have actions of groups that are generated by reflections (Coxeter groups). The basic properties are given in [2]. The rigidity theorem is proved in [6]. In this case, we have to show that the elements that act as reflections in the standard model, act as reflections on N.

For $n = 2$ we have the toric manifolds, which are the non-singular toric varieties and their topological analogue, the quasitoric manifolds ([1], [3]). The result is given in [5]. To show that the action of T^n on N is locally standard, we study the representation s of T^n. To show that N is homeomorphic to the standard model of the action, we show that an element in the local Čech cohomology of the quotient map vanishes.

For the remaining case, we work along the lines of the Coxeter groups and quasitoric varieties. Let $Q^n = (S^3)^n$. We say that Q^n acts on a manifold M^{4n} locally regularly if, locally, the action is given by (quaternionic) multiplication or conjugation on each coordinate. Then the quotient is a manifold with corners. Conversely, starting with a manifold with corners and an appropriate function from its faces to the conjugacy classes of subgroups of Q^n, we can construct a locally regular (quoric) manifold. Our main result is the following.

Theorem 1. *(Rigidity of Quoric Manifolds). Let M^{4n} be a closed locally regular quoric manifold over a nice n-manifold with corners X and X is a homotopy polytope i.e. all the faces of X (and X itself) are contractible manifolds with corners. Let N^{4n} be a locally linear closed Q^n-manifold* *and $f : N^{4n} \longrightarrow M^{4n}$ a Q^n-equivariant homotopy equivalence. Then f is Q^n-homotopic to a Q^n-homeomorphism.*

The proof of the main theorem follows the methods of [6] and [5].

- We show that the action on N^n is locally regular. For this result, we first prove that N^n has the same isotropy groups as M^n and f is an isovariant homotopy equivalence. Then we prove that the action on N^n is locally regular. That is quite different from the torus case. The reason is that this part depends on the representation theory of the underlying group. But Q^n is not abelian and thus its representation theory is more complicated that that of T^n. So, we need a more thorough analysis in this case.
- Let Y be the quotient manifold with corners of the action. We prove that N^n is Q^n-homeomorphic with the standard model constructed from Y. For this, there is an obstruction theory analogous to the torus case.
- The rest is standard. The map f induces a face preserving homotopy equivalence $\phi : Y \longrightarrow X$. Induction and standard surgery methods imply that ϕ is face homotopic to a face homeomorphism ψ. The map ψ induces a Q^n-homeomorphism $g : N^n \longrightarrow M^n$ that is homotopic to f.

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On the multiplicative order convergence on Banach lattice f-algebras

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Let E ,F be Archimedean vector lattices.The Fremlin tensor product $E\overline{\bigotimes}F$ of E and F was introduced by Fremlin in [6]. $E\overline{\bigotimes}F$ contains the algebraic tensor product $E \bigotimes F$ as an ordered vector subspace satisfying density properties. The Fremlin projective tensor product $E\widehat{\bigotimes}F$ of Banach lattices E and F is a Banach lattice, [7]. It contains the Fremlin tensor product $E\overline{\bigotimes}F$ as a norm dense normed lattice. It is known that the Fremlin tensor product $A\overline{\bigotimes}B$ is an f-algebra if A and B are f-algebras, [4,5]. Also, we know that if A and B are Banach lattice f-algebras, then the Fremlin projective tensor product$A\widehat{\bigotimes}B$ of A and B is a Banach lattice f-algebra, [9].

A vector lattice E under an associative multiplication is called a lattice ordered algebra whenever the multiplication makes E an algebra with the usual properties and multiplication of positive elements in E is positive. A lattice ordered algebra A is called an f-algebra if $x \wedge y = 0$ for every $x, y \in A$ implies $(zx) \wedge y = (xz) \wedge y = 0$ for all $z \in A^+$, where A^+ denotes the positive part of A. . A Banach algebra A is called a Banach lattice algebra if A is a Banach lattice and the multiplication of positive elements in A is positive. A Banach lattice algebra A is called a Banach lattice f-algebra if A is an f-algebra.

Definition 1. A net (x_α) in an Archimedean vector lattice E is called order convergent to $x \in E$ if there exists a net (y_β) satisfying $y_\beta \downarrow 0$, and for any β there exists α_β such that $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_\beta$.

Definition 2. Let A be an f-algebra. A net (x_α) in A is called multiplicative order convergent to $x \in A$ if $|x_\alpha - x|.u \to 0$ convergences in order for all $u \in A^+$.

Definition 3. A lattice ordered algebra A is called a normed lattice ordered algebra whenever it is a normed vector lattice and $\|x.y\| \leq \|x\|\|y\|$ holds for all $x, y \in A$.

Definition 4. A net (x_α) in a normed lattice ordered algebra A is called multiplicative norm convergent to $x \in A$ if $\||x_\alpha - x|.u\| \to 0$ for all $u \in A^+$.

The concept of convergence in f-algebras related to multiplication was given before by A. Aydin, [1,2]. The studies under the unbounded order convergence and unbounded norm convergence in vector lattices and Banach lattices were done before by many authors.

O. Zabeti applied the unbounded order convergence to the Fremlin tensor product of vector lattices and the unbounded norm convergence to the Fremlin projective tensor product of Banach lattices in [10].

Our aim is to investigate the multiplicative order convergence in the Fremlin tensor product of two f-algebras and the multiplicative norm convergence in the Fremlin projective tensor product of two Banach lattice f-algebras.

For this subject, we give the following references.

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Octonionic Stiefel manifolds and vector bundles

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The octonions $\mathbb{O}$ satisfy a weaker form of associativity. Namely, they are alternative and power associative only and are not as well known as complex numbers $\mathbb{C}$ and the quaternions $\mathbb{H}$ which are much more widely studied and used.

This talk studies: Stiefel and Grassmann varieties, and vector bundles over octonions $\mathbb{O}$.

Stiefel manifolds. Let $\mathbb{F}$ stand for $\mathbb{R}$, the reals, $\mathbb{C}$, the complex numbers or $\mathbb{H}$, the quaternions.

Recall that the Stiefel manifold $V_{n,r}(\mathbb{F})$ for $r \leq n$ is the set of all orthonormal r-frames in $\mathbb{F}^n$ which can be thought of as a set of $n \times k$ matrices by writing a r-frame as a matrix of k column vectors in $\mathbb{F}^n$. We then have

$$V_{n,r}(\mathbb{F}) = \left\{A \in M_{n,r}(\mathbb{F}) : \bar{A}^t A = I_r\right\}$$

and define

$$V_{n,r}(\mathbb{O}) = \left\{A \in M_{n,r}(\mathbb{O}) : \bar{A}^t A = I_r\right\}.$$

Those yield $V_{n,r}(\mathbb{F})$ and $V_{n,r}(\mathbb{O})$ as algebraic varieties over $\mathbb{R}$ (see [1, 3] for details).

Each $V_{n,r}(\mathbb{F})$ can be viewed as a homogeneous space:

$$V_{n,r}(\mathbb{F}) \cong \mathrm{U}(n,\mathbb{F})/\mathrm{U}(n-r,\mathbb{F}).$$

But, for $V_{n,r}(\mathbb{O})$ we have:

Proposition 1. (1) *$V_{n,r}(\mathbb{O})$ is a compact smooth submanifold of $M_{n,r}(\mathbb{O})$ for any $r \leq n$.*

(2) *$V_{n,r}(\mathbb{O})$ is path-connected for any $r \leq n$ and the map $\pi : V_{n,r+1}(\mathbb{O}) \to V_{n,r}(\mathbb{O})$, given by $\pi(A|v) = A$, is a smooth fibre bundle.*

Grassmann manifolds. Grassmann manifold $G_{n,r}(\mathbb{F})$ is a differentiable manifold that parameterizes the set of all r-dimensional linear subspaces of $\mathbb{F}^n$. Since the rank of an orthogonal projection operator equals its trace, we can identify

$$G_{n,r}(\mathbb{F}) = \left\{A \in M_n(\mathbb{F}) : A = \bar{A}^t = A^2, \operatorname{tr}(A) = r\right\}$$

and define

$$G_{n,r}(\mathbb{O}) = \left\{A \in M_n(\mathbb{O}) : A = \bar{A}^t = A^2, \operatorname{tr}(A) = r\right\}.$$

Those yield $G_{n,r}(\mathbb{F})$ and $G_{n,r}(\mathbb{O})$ as algebraic varieties over $\mathbb{R}$ (see [1, 3] for details).

Each $G_{n,r}(\mathbb{F})$ can be viewed as a homogeneous space:

$$G_{n,r}(\mathbb{F}) \cong \mathrm{U}(n,\mathbb{F})/\mathrm{U}(r,\mathbb{F}) \times \mathrm{U}(n-r,\mathbb{F}).$$

Furthermore, we have the principal $\mathrm{U}(r)$-bundle

$$\mathrm{U}(r,\mathbb{F}) \hookrightarrow V_{n,r}(\mathbb{F}) \to G_{n,r}(\mathbb{F})$$

for the Stiefel map $V_{n,r}(\mathbb{F}) \to G_{n,r}(\mathbb{F})$.

Due to the non-associativity of $\mathbb{O}$ we do not have a Stiefel map $\pi : V_{n,r}(\mathbb{O}) \to G_{n,r}(\mathbb{O})$, but we may define a subset $V'_{n,r}(\mathbb{O}) \subseteq V_{n,r}(\mathbb{O})$ as follows: $A \in V'_{n,r}(\mathbb{O})$ if the set of all entries of A

generate an associative subalgebra of $\mathbb{O}$. Then, we have a Stiefel map $\pi : V'_{n,r}(\mathbb{O}) \to G_{n,r}(\mathbb{O})$ given by $\pi(A) = A\bar{A}^t$.

Similarly to $V'_{n,r}(\mathbb{O})$, we define $G'_{n,r}(\mathbb{O})$ as follows: $A \in G'_{n,r}(\mathbb{O})$ if all entries of A generate an associative subalgebra of $\mathbb{O}$. It is clear that the Stiefel map $\pi : V'_{n,r}(\mathbb{O}) \to G_{n,r}(\mathbb{O})$ yields a surjective map $\pi : V'_{n,r}(\mathbb{O}) \to G'_{n,r}(\mathbb{O})$. In particular, $G'_{n,r}(\mathbb{O})$ is piecewise-smooth path-connected.

Vector bundles

By analogy with real, complex or quaterionic vector bundles, define an octonionic vector bundle of rank r over some space X as a continuous map

$$f : X \to G'_{-,r}(\mathbb{O}).$$

The following result holds by adapting proof of [1, Theorem 12.1.7] or [2, Theorem 2.5]:

Theorem 2. *Any smooth map $X \to G'_{n,r}(\mathbb{O})$ with $r \leq n$ and X, a compact non-singular algebraic set, is homotopic to an entire rational map from X to $G'_{n,r}(\mathbb{O})$.*

Then, we derive:

Corollary 3. *Any smooth map $X \to G_{n,r}(\mathbb{O})$ with $n = 2, 3$ and $r = 1$ is homotopic to an entire rational map $X \to G_{n,r}(\mathbb{O})$. In particular, any continuous map $\mathbb{S}^n \to \mathbb{S}^8$ is homotopic to an entire rational map $\mathbb{S}^n \to \mathbb{S}^8$.*

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Some series involving central binomial coefficients

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Using Maclaurin expansion $\arcsin = \sum\limits_{n=0}^{\infty} \frac{1}{2^{2n}}\binom{2n}{n}\frac{x^{2n+1}}{2n+1}$ and, for non-zero real variable x, formulas

$$\Re\Big(\arcsin\big(x\sqrt{i}\,\big)\Big) = \arctan\sqrt{\frac{\sqrt{1+x^4}-1}{x^2}},$$

$$\Im\Big(\arcsin\big(x\sqrt{i}\,\big)\Big) = \operatorname{arctanh}\sqrt{\frac{\sqrt{1+x^4}-1}{x^2}},$$

we obtain some series involving central binomial coefficients $\binom{2n}{n}$; see [1] for more details.

Theorem 1. *For $|x| \leq 1$, we have*

$$\sum_{n=0}^{\infty} \frac{(-1)^{\lfloor 3n/2 \rfloor}}{4^n(2n+1)}\binom{2n}{n}x^{2n+1} = \sqrt{2}\arctan\Big(\frac{\sqrt{\sqrt{1+x^4}-1}}{x}\Big),$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}}{4^n}\binom{2n}{n}x^{2n}=\frac{\sqrt{2\sqrt{1+x^4}-2}}{\sqrt{1+x^4}\big(x^2-1+\sqrt{1+x^4}\big)},$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}n}{4^n}\binom{2n}{n}x^{2n}=\frac{x^2}{\sqrt{2}}\cdot\frac{3x^6-4x^4+5x^2-2+(3x^4-5x^2+2)\sqrt{1+x^4}}{(\sqrt{1+x^4}+x^2-1)^2\sqrt{(1+x^4)^3}\sqrt{\sqrt{1+x^4}-1}}.$$

Example 2. If $x=1$, $x=\sqrt{2}/2$, and $x=1/2$ then from Theorem 1 we have

$$\sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}}{4^n(2n+1)}\binom{2n}{n}=\sqrt{2}\operatorname{arccot}\sqrt{\delta},\qquad \sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}}{4^n}\binom{2n}{n}=\frac{1}{\sqrt{2\delta}},$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}n}{4^n}\binom{2n}{n}=-\frac{\sqrt{\delta}}{4};$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}}{8^n(2n+1)}\binom{2n}{n}=2\operatorname{arccot}\big(\alpha\sqrt{\alpha}\big),\qquad \sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}}{8^n}\binom{2n}{n}=\frac{2}{\sqrt{5\alpha}},$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}n}{8^n}\binom{2n}{n}=-\frac{\sqrt{5}}{25}\alpha^2\sqrt{\alpha};$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}}{16^n(2n+1)}\binom{2n}{n}=2\sqrt{2}\operatorname{arccot}\sqrt{\sqrt{17}+4},\qquad \sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}}{16^n}\binom{2n}{n}=\frac{2}{\sqrt{17}}\sqrt{\sqrt{17}-1},$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}n}{16^n}\binom{2n}{n}=-\frac{1}{17\sqrt{17}}\sqrt{17\sqrt{17}+47},$$

where $\alpha=(1+\sqrt{5})/2$ and $\delta=\sqrt{2}+1$ are the golden and silver ratios, respectively.

We will also establish connections with the Fibonacci and Lucas numbers. As usual, the Fibonacci numbers F_n and the Lucas numbers L_n are defined, for $n\in\mathbb{Z}$, through the recurrence $F_n=F_{n-1}+F_{n-2}$, $n\geq 2$, with initial values $F_0=0$, $F_1=1$ and $L_n=L_{n-1}+L_{n-2}$ with $L_0=2$, $L_1=1$. For negative subscripts, we have $F_{-n}=(-1)^{n-1}F_n$ and $L_{-n}=(-1)^nL_n$.

Theorem 3. *For any integer s,*

$$\sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}}{16^n(2n+1)}\binom{2n}{n}F_{2n+s}=-\frac{2\sqrt{10}}{5}\big(\alpha^{s-1}\arctan C_1-\beta^{s-1}\arctan D_1\big),$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}}{16^n(2n+1)}\binom{2n}{n}L_{2n+s}=-2\sqrt{2}\big(\alpha^{s-1}\arctan C_1+\beta^{s-1}\arctan D_1\big);$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}}{16^n}\binom{2n}{n}F_{2n+s}=\frac{8\sqrt{205}}{615}\,(\alpha^sC_2+\beta^sD_2),$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}}{16^n}\binom{2n}{n}L_{2n+s}=\frac{40\sqrt{41}}{615}\,(\alpha^sC_2-\beta^sD_2);$$

$$\sum_{n=0}^{\infty}(-1)^{\lfloor 3n/2\rfloor}\frac{n}{16^n}\binom{2n}{n}F_{2n+s}=\frac{\sqrt{205}}{226935}\big(\alpha^{s+2}C_3-\beta^{s+2}D_3\big),$$

$$\sum_{n=0}^{\infty}(-1)^{\lfloor 3n/2\rfloor}\frac{n}{16^n}\binom{2n}{n}L_{2n+s}=\frac{\sqrt{41}}{45387}\big(\alpha^{s+2}C_3+\beta^{s+2}D_3\big),$$

where

$$C_1 = \beta\sqrt{\frac{\sqrt{78+6\sqrt{5}}-8}{2}}, \quad D_1 = \alpha\sqrt{\frac{\sqrt{78-6\sqrt{5}}-8}{2}},$$

$$C_2 = \frac{\sqrt{\sqrt{78+6\sqrt{5}}-8}\,\sqrt{78-6\sqrt{5}}}{-5+\sqrt{5}+\sqrt{78+6\sqrt{5}}}, \quad D_2 = \frac{\sqrt{\sqrt{78-6\sqrt{5}}-8}\,\sqrt{78+6\sqrt{5}}}{5+\sqrt{5}-\sqrt{78-6\sqrt{5}}},$$

$$C_3 = \frac{\big(148-112\sqrt{5}-\sqrt{78+6\sqrt{5}}\,(25-11\sqrt{5})\big)\sqrt{(78-6\sqrt{5})^3}}{\big(-5+\sqrt{5}+\sqrt{78+6\sqrt{5}}\big)^2\sqrt{\sqrt{78+6\sqrt{5}}-8}},$$

$$D_3 = \frac{\big(148+112\sqrt{5}-\sqrt{78-6\sqrt{5}}\,(25+11\sqrt{5})\big)\sqrt{(78+6\sqrt{5})^3}}{\big(5+\sqrt{5}-\sqrt{78+6\sqrt{5}}\big)^2\sqrt{\sqrt{78-6\sqrt{5}}-8}}.$$

Note that since $\binom{2n}{n} = (n+1)C_n$, where C_n are Catalan numbers, our results could be stated equivalently in terms of the Catalan numbers. Similar series were studied recently in [2, 3, 4, 5].

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On non-topologizable semigroups

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In this paper we shall follow the semigroup terminology of [1, 2, 3, 4].

Throughout these abstract we always assume that all topological spaces involved are Hausdorff — unless explicitly stated otherwise.

Definition 1. Let X, Y and Z be topological spaces. A map $f\colon X\times Y\to Z$, $(x,y)\mapsto f(x,y)$, is called

- (i) *right* [*left*] *continuous* if it is continuous in the right [left] variable; i.e., for every fixed $x_0\in X$ [$y_0\in Y$] the map $Y\to Z$, $y\mapsto f(x_0,y)$ [$X\to Z$, $x\mapsto f(x,y_0)$] is continuous;
- (ii) *separately continuous* if it is both left and right continuous;
- (iii) *jointly continuous* if it is continuous as a map between the product space $X\times Y$ and the space Z.

Definition 2. Let S be a non-void topological space which is provided with an associative multiplication (a semigroup operation) $\mu\colon S\times S\to S$, $(x,y)\mapsto \mu(x,y)=xy$. Then the pair (S,μ) is called

(*i*) a *right topological semigroup* if the map μ is right continuous, i.e., all interior left shifts $\lambda_s\colon S\to S$, $x\mapsto sx$, are continuous maps, $s\in S$;
(*ii*) a *left topological semigroup* if the map μ is left continuous, i.e., all interior right shifts $\rho_s\colon S\to S$, $x\mapsto xs$, are continuous maps, $s\in S$;
(*iii*) a *semitopological semigroup* if the map μ is separately continuous;
(*iv*) a *topological semigroup* if the map μ is jointly continuous.

We usually omit the reference to μ and write simply S instead of (S,μ). It goes without saying that every topological semigroup is also semitopological and every semitopological semigroup is both a right and left topological semigroup.

A topology τ on a semigroup S is called:

- a *semigroup* topology if (S,τ) is a topological semigroup;
- a *shift-continuous* topology if (S,τ) is a semitopological semigroup;
- an *left-continuous* topology if (S,τ) is a left topological semigroup;
- an *right-continuous* topology if (S,τ) is a right topological semigroup.

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq=1$. The semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$q^kp^l\cdot q^mp^n=\begin{cases} q^{k-l+m}p^n, & \text{if } l<m;\\ q^kp^n, & \text{if } l=m;\\ q^kp^{l-m+n}, & \text{if } l>m.\end{cases}$$

We define the following subsets of the bicyclic monoid

$$\mathscr{C}_+(p,q)=\left\{q^ip^j\in\mathscr{C}(p,q)\colon i\leqslant j\right\}\quad\text{and}\quad \mathscr{C}_-(p,q)=\left\{q^ip^j\in\mathscr{C}(p,q)\colon i\geqslant j\right\}.$$

Proposition 3. *$\mathscr{C}_+(p,q)$ and $\mathscr{C}_-(p,q)$ are anti-isomorphic submonoids of $\mathscr{C}(p,q)$.*

Proposition 4. *Green's relations $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{D}$ and $\mathscr{H}$ on monoids $\mathscr{C}_+(p,q)$ and $\mathscr{C}_-(p,q)$ coincide with the equality relation.*

Theorem 5. *Every Hausdorff left-continuous topology on the monoid $\mathscr{C}_+(p,q)$ is discrete.*

Theorem 6. *Every Hausdorff right-continuous topology on the monoid $\mathscr{C}_-(p,q)$ is discrete.*

Example 7. There exists a non-discrete locally compact semigroup T_1-topology τ on the monoid $\mathscr{C}_+(p,q)$.

Example 8. There exists a non-discrete compact shift-continuous T_1-topology τ on the monoid $\mathscr{C}_+(p,q)$.

Proposition 9. *If the monoid $\mathscr{C}_+(p,q)$ is a dense subsemigroup of a Hausdorff semitopological monoid S and $I=S\setminus\mathscr{C}_+(p,q)\neq\varnothing$ then I is a closed two-sided ideal of the semigroup S.*

Example 10. There exists a compact Hausdorff topological monoid S which contains the monoid $\mathscr{C}_+(p,q)$ as a dense submonoid.

Also, we discuss under which conditions a shift-continuous T_1-topology τ on the monoid $\mathscr{C}_+(p,q)$ is discrete.

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Existence and non-existence of cohomogeneity one Einstein metrics

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A Riemannian metric g is Einstein if $\mathrm{Ric}(g) = \Lambda g$ for some constant Λ. A general existence theorem for homogeneous Einstein metrics was established in [WZ86]. It is natural to turn to the cohomogeneity one Einstein metrics, meaning that the principal orbit G/K is of codimension one. The cohomogeneity one condition reduces the Einstein equation to a system of ODEs. Previously known examples include [Pag78], [BB82], [KS86], [KS88], and [WW98]. Recently, we proved the existence of an Einstein metric on $\mathbb{HP}^{m+1}\sharp\overline{\mathbb{HP}}^{m+1}$ [Chi24], generalizing the result in [Böh98] to all higher dimensions.

We realize that the analytic techniques can be carried over to many other cohomogeneity one spaces. We develop two criteria to check the existence or non-existence of a cohomogeneity one Einstein metrics with a certain fixed principal orbit type. In particular, the principal orbit G/K is the total space of a sphere bundle over a singular orbit G/K, and both the fiber and the base space are irreducible. Each such a principal orbit is associated to a structural triple (d_1, d_2, A), where $d_1 = \dim(\mathsf{H}/\mathsf{K})$, $d_2 = \dim(\mathsf{G}/\mathsf{H})$ and $A > 0$ is a constant obtained from the O'neil tensor in the theory of Riemannian submersion. The corresponding cohomogeneity one space, denoted as M, is a double disk bundle, where G/K collapses to G/H on two ends. The Einstein metric is obtained from the ansatz

$$dt^2 + f_1^2(t)\, b|_{\mathfrak{h}/\mathfrak{k}} + f_2^2(t)\, b|_{\mathfrak{g}/\mathfrak{h}}\,, \tag{1}$$

where t parametrizes the 1-dimensional orbit space and b is a background metric.

Our existence theorem is the following.

Theorem 1. *For any (d_1, d_2) with $d_2 \geq d_1 \geq 2$, there exists a constant $\chi_{d_1,d_2} \in \left(0, \frac{d_2(d_2-1)^2}{d_1^2(d_1d_2-d_2+4)}\right]$ such that if G/K is a principal orbit with $A \in [0, \chi_{d_1,d_2})$, then there is at least one cohomogeneity one Einstein metrics on M.*

The constant χ_{d_1,d_2} is an algebraic function in (d_1, d_2), whose formula is very complicated in general. Nevertheless, we obtain many new examples of inhomogeneous Einstein metrics from previous works on homogeneous Einstein metrics including [DZ79], [WZ85], [Wan92], [DK08], [Nik16], [PZ21], and [LW24].

On the other hand, we also have the following non-existence theorem.

Theorem 2. *Define*

$$\Psi_{d_1,d_2} := \frac{(4(d_1-1)n^2 + d_2^2)(3n + d_1)}{(2n^2 + n + d_1)^2 d_1^2}\,\frac{d_2(d_2-1)^2}{4(d_1-1)}.$$

If G/K is a principal orbit with $(d_1, d_2) \notin \{(2, 2), (2, 3), (2, 4)\}$ and $A \geq \Psi_{d_1,d_2}$, then there does not exists any G-invariant cohomogeneity one Einstein metrics on M from ansatz (1).

We find some examples of Theorem 2 from the classification in [DK08], including $\mathbb{OP}^2\sharp\overline{\mathbb{OP}}^2$ with Spin(9)/Spin(7) as its principal orbit.

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Stability of vertical minimal surfaces in three-dimensional sub-Riemannian manifolds

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A *sub-Riemannian manifold* is a smooth manifold M together with a completely non-integrable smooth distribution $\mathcal{H}$ on M (it is called a *horizontal distribution*) and a smooth field of Euclidean scalar products $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ on $\mathcal{H}$ (it is called a *sub-Riemannian metric*). In particular, $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ can be constructed as a restriction of some Riemannian metrics $\langle\cdot,\cdot\rangle$ on M to $\mathcal{H}$. Here we will assume that all sub-Riemannian structures are of this form. Let Σ be a smooth oriented surface in a three-dimensional sub-Riemannian manifold M. If N_h is the orthogonal projection of the unit normal field N of Σ (in the Riemannian sense) onto $\mathcal{H}$ and $d\Sigma$ is the Riemannian area form of Σ, then the *sub-Riemannian area* of a domain $D \subset \Sigma$ is defined as $A(D) = \int\limits_D |N_h|\, d\Sigma$. The *normal variation* of the surface Σ defined by a smooth function u is the map $\varphi\colon \Sigma \times I \to M\colon \varphi_s(p) = \exp_p(su(p)N(p))$, where I is an open neighborhood of 0 in $\mathbb{R}$ and $\exp_p$ is the Riemannian exponential map in p. Denote $A(s) = \int\limits_{\Sigma_s} |N_h|\, d\Sigma_s$, where $\Sigma_s = \varphi_s(\Sigma)$. Then $A'(0)$ is called the *first (normal) area variation* defined by φ, and $A''(0)$ is called the *second* one. A surface Σ is called *minimal* if $A'(0) = 0$ for any normal variations with compact support in $\Sigma \setminus \Sigma_0$, where $\Sigma_0 = \{p \in \Sigma \mid N_h(p) = 0\}$ is the *singular set* of Σ. A minimal surface Σ is called *stable* if $A''(0) \geq 0$ for any normal variations with compact support in $\Sigma \setminus \Sigma_0$. We will call a surface Σ in a three-dimensional sub-Riemannian manifold *vertical* if $T_p\Sigma \perp \mathcal{H}_p$ for each $p \in \Sigma$. In particular, for such surfaces $N_h = N$ and $\Sigma_0 = \varnothing$.

Proposition 1. *Let* Σ *be an oriented vertical surface in a three-dimensional sub-Riemannian manifold* M*. Then its first normal area variation defined by a smooth function* u *with compact support equals* $A'(0) = -\int_\Sigma 2Hu\, d\Sigma$*, where* H *is the Riemannian mean curvature of* Σ*. Thus,* Σ *is minimal in the sub-Riemannian sense if and only if it is minimal in the Riemannian sense.*

Proposition 2. *Let* Σ *be an oriented vertical minimal surface in a three-dimensional sub-Riemannian manifold* M*. Then its second normal area variation defined by a smooth function* u *with compact support equals*

$$A''(0) = \int_\Sigma -\left(X(u) - \langle \nabla_N X, N\rangle u\right)^2 + |\nabla_\Sigma u|^2 - \left(\mathrm{Ric}\,(N,N) + |B|^2\right) u^2\, d\Sigma,$$

where ∇ *and* Ric *are the Riemannian connection and the Ricci tensor of* M *respectively,* X *is the unit normal vector field of* $\mathcal{H}$ *(which is tangent to* Σ *because it is vertical),* ∇_Σ *and* B *are the Riemannian gradient and the second fundamental form of* Σ *respectively. It follows that if* Σ *is stable in the sub-Riemannian sense, it is also stable in the Riemannian sense.*

Let us discuss some examples of such surfaces. In all these examples M is a Lie group, $\mathcal{H}$ and $\langle\cdot,\cdot\rangle$ are left-invariant. In [2] it was shown that a complete connected minimal surface with the empty singular set (in particular, vertical) in the sub-Riemannian three-dimensional Heisenberg group is stable if and only if it is a vertical Euclidean plane. In [3] the authors considered the standard three-dimensional sphere with the horizontal distribution defined by the Hopf field X and showed that complete connected vertical minimal surfaces are Clifford tori. It is well-known that they are not stable in the Riemannian sense, hence also in the sub-Riemannian sense. In [1] we proved that in the solvable Lie group $\widetilde{E(2)}$, which is the universal covering of the proper motions group of the Euclidean plane, with the Euclidean metric and a left-invariant horizontal distribution all complete connected vertical minimal surfaces are Euclidean planes and standard helicoids. We showed that planes are stable in the sub-Riemannian sense, and it is known that helicoids are not stable in the Riemannian sense, hence also in the sub-Riemannian sense.

The three-dimensional Thurston geometry Sol is the space $\mathbb{R}^3$ with coordinates (x, y, z) and with the following orthonormal basis of left-invariant vector fields defined by its solvable Lie group structure:

$$X_1 = \frac{1}{\sqrt{2}}\left(e^{-z}\frac{\partial}{\partial x} + e^{z}\frac{\partial}{\partial y}\right),\ X_2 = \frac{1}{\sqrt{2}}\left(e^{-z}\frac{\partial}{\partial x} - e^{z}\frac{\partial}{\partial y}\right),\ X_3 = \frac{\partial}{\partial z}.$$

Note that $[X_2, X_3] = X_1$, so the left-invariant distribution $\mathcal{H}$ orthogonal to X_1 is completely non-integrable. Let us consider a sub-Riemannian structure on Sol such that $\mathcal{H}$ is horizontal. It then follows from the results of [4] that any complete connected vertical minimal surface in Sol after an isometry becomes either a Euclidean plane $z = C$ or a "helicoid"

$$(s, t) \mapsto \left(\frac{1}{\sqrt{2}}e^{-t}s + C_1, \frac{1}{\sqrt{2}}e^{t}s + C_2, t\right).$$

Using this description, we are able to prove the following.

Proposition 3. *All vertical minimal surfaces in Sol are stable in the sub-Riemannian sense and thus in the Riemannian sense.*

The three-dimensional Thurston geometry $\widetilde{SL(2,\mathbb{R})}$ can be described as the universal covering of the unit tangent bundle of the hyperbolic plane H^2 with the Sasaki metric, that is, the half-space $\{(x, y, z) \in \mathbb{R}^3 \mid y > 0\}$ with the following orthonormal basis of left-invariant vector fields with respect to its simple Lie group structure:

$$X_1 = y\left(-\sin z\frac{\partial}{\partial x} + \cos z\frac{\partial}{\partial y}\right) + \sin z\frac{\partial}{\partial z},\ X_2 = y\left(-\cos z\frac{\partial}{\partial x} - \sin z\frac{\partial}{\partial y}\right) + \cos z\frac{\partial}{\partial z},\ X_3 = \frac{\partial}{\partial z}.$$

In particular, $[X_1, X_2] = -X_3$, so the left-invariant distribution $\mathcal{H}$ orthogonal to X_3 is completely non-integrable. Consider a sub-Riemannian structure on this manifold such that $\mathcal{H}$ is horizontal. We than obtain the following description.

Theorem 4. *Any complete connected vertical minimal surface in* $\widetilde{SL(2,\mathbb{R})}$ *has either the parameterization* $(s, t) \mapsto (C, s, t)$ *or* $(s, t) \mapsto \left(C_1 + \frac{1}{C_2}\sin C_2 s, -\frac{1}{C_2}\cos C_2 s, t\right)$ *and so is a cylinder over a geodesic in* H^2*. All vertical minimal surfaces in* $\widetilde{SL(2,\mathbb{R})}$ *are stable in the sub-Riemannian sense and thus in the Riemannian sense.*

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Rational factorization of Lax type flows in the space dual to the centrally extended Lie algebra of matrix super-integro-differential operators

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In papers of A. Prykarpatski, O. Hentosh and their coauthors the Lie-algebraic approach to the rational factorization of Lax type flows in spaces dual to certain operator Lie algebras and central extensions of some of them is developed for the central extension of the Lie algebra [1] of matrix super-integro-differential operators with one anticommuting variable.

Let us consider the Lie algebra g which consists of matrix super-integro-differential operators such as $\mathcal{A} := \mathbf{1}\partial^q + \sum_{p<2q} A_p D_\theta^p$, where $A_r \in C^\infty(\mathbb{S}\times\Lambda_1; gl(m|n))$, $gl(m|n))$ is a semi-simple Lie superalgebra of square supermatrices, $A_p = A_p(x,\theta) := A_p^0(x) + \theta A_p^1(x)$, $p \in \mathbb{Z}$, the supermatrices A_p are even for every even p and odd for every odd p, $\mathbf{1} \in gl(m+n)$ is an unit matrix, $q \in \mathbb{N}$, $\partial = \partial/\partial x$, $x \in \mathbb{S} \simeq \mathbb{R}/2\pi\mathbb{Z}$, $\theta \in \Lambda_1$, $\Lambda := \Lambda_0 \oplus \Lambda_1$ is a Grassmann algebra over the field $\mathbb{C} \subset \Lambda_0$, $D_\theta := \partial/\partial\theta + \theta\partial/\partial x$ is a superderivative, for which $D_\theta^2 = \partial/\partial x$, and $\partial/\partial\theta$ is a left partial derivative by the anticommuting variable θ, with the standard commutator $[.,.]$, acting by the rule:

$$[\mathcal{A},\,\mathcal{B}] = \mathcal{A}\circ\mathcal{B} - \mathcal{B}\circ\mathcal{A}, \quad \mathcal{A},\mathcal{B} \in \mathrm{g}, \tag{1}$$

where symbol "$\circ$" denotes the product of two matrix super-integro-differential operators (see [1]). The scalar product $(\mathcal{A},\,\mathcal{B}) = \int_{x\in\mathbb{S}} dx \int d\theta\ \mathrm{sSp}\,\mathrm{res}_{D_\theta}(\mathcal{A}\mathcal{B})$, where "$\mathrm{res}_{D_\theta}$" denotes the coefficient at D_θ^{-1} in the expansion of a matrix super-integro-differential operator and "sSp" is a supermatrix supertrace, being invariant with respect to the commutator (1), allows us to identify the dual space g^* to g, with the Lie algebra itself. The latter is splitting into the direct sum $\mathrm{g} := \mathrm{g}_+ \oplus \mathrm{g}_-$ of two its Lie subalgebras, where g_+ is a Lie subalgebra of formal polynomials by the superderivative operator with supermatrix-valued coefficients, and $\mathrm{g}_+^* \simeq \mathrm{g}_-$, $\mathrm{g}_-^* \simeq \mathrm{g}_+$.

One constructs the central extension $\hat{\mathrm{g}} := \tilde{\mathrm{g}} \oplus \mathbb{C}$ of parameterized Lie algebra $\tilde{\mathrm{g}} := \prod_{y\in\mathbb{S}} \mathrm{g}$ by the Maurer-Cartan two-cocycle on $\tilde{\mathrm{g}}$ such as $\omega_2(\mathcal{A},\mathcal{B}) = \int_{y\in\mathbb{S}} dy\,(\mathcal{A}, \partial\mathcal{B}/\partial y)$, where $\mathcal{A},\mathcal{B} \in \tilde{\mathrm{g}}$, with the commutator

$$[(\mathcal{A},d),\,(\mathcal{B},e)] = ([\mathcal{A},\mathcal{B}],\,\omega_2(\mathcal{A},\mathcal{B})), \quad (\mathcal{A},d),(\mathcal{B},e) \in \hat{\mathrm{g}},$$

and introduces another commutator on $\hat{\mathrm{g}}$ in the form

$$\begin{gathered}[(\mathcal{A},d),\,(\mathcal{B},e)]_{\mathcal{R}} = ([\mathcal{A},\mathcal{B}]_{\mathcal{R}},\,\omega_{2,\mathcal{R}}(\mathcal{A},\mathcal{B})), \\ [\mathcal{A},\mathcal{B}]_{\mathcal{R}} = [\mathcal{R}\mathcal{A},\mathcal{B}] + [\mathcal{A},\mathcal{R}\mathcal{B}], \quad \omega_{2,\mathcal{R}}(\mathcal{A},\mathcal{B}) = \omega_2(\mathcal{R}\mathcal{A},\mathcal{B}) + \omega_2(\mathcal{A},\mathcal{R}\mathcal{B}),\end{gathered} \tag{2}$$

where $\mathcal{R} = (P_+ - P_-)/2$ and $P_\pm$ are projectors on the Lie subalgebras $\tilde{\mathrm{g}}_\pm$. On the space $\hat{\mathrm{g}}^*$, dual to $\hat{\mathrm{g}}$ with respect to the scalar product $((\mathcal{A},d),\,(\mathcal{B},e)) = \int_{y\in\mathbb{S}} dy\,(\mathcal{A},\mathcal{B}) + de$, the commutator (2) determines the Lie-Poisson bracket

$$\{\gamma,\mu\}_{\mathcal{R}}(l) = \int_{y\in\mathbb{S}} dy\,(l,\,[\nabla_l\gamma(l),\nabla_r\mu(l)]_{\mathcal{R}}) + c\omega_2(\nabla_l\gamma(l),\nabla_r\mu(l)) = (\nabla_l\gamma(l),\,\Theta\nabla_r\mu(l)), \tag{3}$$

where $\gamma,\mu \in \mathcal{D}(\tilde{\mathrm{g}}^*)$ are arbitrary smooth by Frechet functionals on $\tilde{\mathrm{g}}^* \simeq \tilde{\mathrm{g}}$, at a point $(l,c) \in \hat{\mathrm{g}}^*$. Here $l \in \tilde{\mathrm{g}}^*$ is some matrix super-integro-differential operator of order $q \in \mathbb{N}$, $c \in \mathbb{C}$, ∇_l, ∇_r are left and right gradient operators accordingly, $\Theta : T^*(\tilde{\mathrm{g}}^*) \to T(\tilde{\mathrm{g}}^*)$ is the Poisson operator generating the Lie-Poisson bracket (3) at a point $l \in \tilde{\mathrm{g}}^*$ and acting as $\Theta : \nabla\gamma(l) \mapsto -[l - c\mathbf{1}\partial/\partial y,\,(\nabla\gamma(l))_-] + [l - c\mathbf{1}\partial/\partial y,\,\nabla\gamma(l)]_-$ for any $\gamma \in \mathcal{D}(\tilde{\mathrm{g}}^*)$, the subscript "$-$" denotes the projection of the corresponding element from $\tilde{\mathrm{g}}$ on the Lie subalgebra $\tilde{\mathrm{g}}_- := \prod_{y\in\mathbb{S}} \mathrm{g}_-$, and $T(\tilde{\mathrm{g}}^*), T^*(\tilde{\mathrm{g}}^*)$ are tangent and cotangent spaces to $\tilde{\mathrm{g}}^*$.

The Casimir functionals $\gamma_j \in \mathcal{I}(\hat{\mathrm{g}}^*)$, $j \in \mathbb{N}$, of the central extension $\hat{\mathrm{g}}$, whose left gradients obey the equalities such that $[l - c\mathbf{1}\partial/\partial y,\ \nabla_l\gamma_j(l)] = 0$, where $\nabla_l\gamma_j(l) := \mathbf{1}\partial^j + \sum_{p<2j} A_{j,p}D_\theta^p$, $A_{j,p}$ are supermatrix-valued functions of suitable parity, $j \in \mathbb{N}$, $p \in \mathbb{Z}$, $p < 2j$, at a point $(l, c) \in \hat{\mathrm{g}}^*$, and the $\mathcal{R}$-deformed Lie-Poisson bracket (3) give us the hierarchy of Lax type Hamiltonian flows on $\tilde{\mathrm{g}}^* \simeq \tilde{\mathrm{g}}$:

$$dl/dt_j = [(\nabla_l\gamma_j(l)))_+,\ l - c\mathbf{1}\partial/\partial y], \quad j \in \mathbb{N}, \quad t_j \in \mathbb{R},$$

where the subscript "+" denotes the projection of the corresponding element from $\tilde{\mathrm{g}}$ on the Lie subalgebra $\tilde{\mathrm{g}}_+ := \prod_{y\in\mathbb{S}} \mathrm{g}_+$. One considers another hierarchy of Lax type Hamiltonian flows on the dual space $\tilde{\mathrm{g}}^*$:

$$d\tilde{l}/dt_j = [(\nabla_l\gamma_j(\tilde{l}))_+,\ \tilde{l} - c\mathbf{1}\partial/\partial y], \quad j \in \mathbb{N}, \quad t_j \in \mathbb{R},$$

for some matrix super-integro-differential operator $\tilde{l} \in \tilde{\mathrm{g}}^*$ of order $q \in \mathbb{N}$, which is related with the operator $l \in \tilde{\mathrm{g}}^*$ by the generalized gauge transformation

$$\tilde{l}(0) - c\mathbf{1}\partial/\partial y = \mathcal{B}(0)^{-1}(l(0) - c\mathbf{1}\partial/\partial y)\mathcal{B}(0), \tag{4}$$

where $\mathcal{B}(0) \in \tilde{\mathrm{g}}_+$ is a matrix superdifferential operator of order $s \in \mathbb{N}$ with constant coefficients, at the initial moment of the time $t_j \in \mathbb{R}$ for every $j \in \mathbb{N}$.

Theorem 1. *If for every $j \in \mathbb{N}$ at the initial moment of the time $t_j \in \mathbb{R}$ matrix super-integro-differential operators $l, \tilde{l} \in \tilde{\mathrm{g}}^*$ of order $q \in \mathbb{N}$ are related by the relationship (4), there exist such matrix superdifferential operators of orders $q + s$ and s accordingly, where $s \in \mathbb{Z}_+$, $s < q$, that the equalities*

$$l = \mathcal{A}\mathcal{B}^{-1}, \quad \tilde{l} = \mathcal{B}^{-1}(\mathcal{A} - c\partial\mathcal{B}/\partial y) \tag{5}$$

hold. The operators $\mathcal{A}, \mathcal{B} \in \tilde{\mathrm{g}}_+$ satisfy the following systems of evolution equations

$$\begin{aligned} d\mathcal{A}/dt_j &= (\nabla_l\gamma_j(l))_+\mathcal{A} - \mathcal{A}(\nabla_l\gamma_j(\tilde{l}))_+ - c(\partial(\nabla_l\gamma_j(l))_+/\partial y)\mathcal{B}, \\ d\mathcal{B}/dt_j &= (\nabla_l\gamma_j(l))_+\mathcal{B} - \mathcal{B}(\nabla_l\gamma_j(\tilde{l}))_+, \quad j \in \mathbb{N}, \end{aligned} \tag{6}$$

which possess an infinite sequence of the conservation laws $H_j \in \mathcal{D}(\tilde{\mathrm{g}}_+ \times \tilde{\mathrm{g}}_+)$, $j \in \mathbb{N}$, in the form

$$H_j(\mathcal{A}, \mathcal{B}) := \gamma_j(l)|_{l=\mathcal{A}\mathcal{B}^{-1}} = \gamma_j(\tilde{l})|_{\tilde{l}=\mathcal{B}^{-1}(\mathcal{A}-c\partial\mathcal{B}/\partial y)}.$$

The equalities (5) determine the Backlund transformation

$$P : (\mathcal{A}, \mathcal{B}) \in \tilde{\mathrm{g}} \times \tilde{\mathrm{g}} \mapsto (l, \tilde{l}) \in \tilde{\mathrm{g}}^* \oplus \tilde{\mathrm{g}}^*. \tag{7}$$

Theorem 2. *For every $j \in \mathbb{N}$ the system of evolution equations (6), given on the subspace $\tilde{\mathrm{g}}_+ \times \tilde{\mathrm{g}}_+ \subset \tilde{\mathrm{g}} \times \tilde{\mathrm{g}}$, is Hamiltonian with respect to the Poisson bracket $\{.,.\}_{\mathcal{L}}$, which arises as a reduction of the Poisson bracket $\{.,.\}_{\bar{\mathcal{L}}}$ with the corresponding Poisson operator $\bar{\mathcal{L}} : T^*(\tilde{\mathrm{g}} \times \tilde{\mathrm{g}}) \to T(\tilde{\mathrm{g}} \times \tilde{\mathrm{g}})$ such as $\bar{\mathcal{L}} = (P')^{-1}(\Theta \oplus \tilde{\Theta})(P'^*)^{-1}$, where $\Theta, \tilde{\Theta} : T^*(\tilde{\mathrm{g}} \times \tilde{\mathrm{g}}) \to T(\tilde{\mathrm{g}} \times \tilde{\mathrm{g}})$ are Poisson operators generating the Lie-Poisson bracket $\{.,.\}_{\mathcal{R}}$ at points $l, \tilde{l} \in \tilde{\mathrm{g}}^*$ accordingly, $P'^* : T^*(\tilde{\mathrm{g}}^* \oplus \tilde{\mathrm{g}}^*) \to T^*(\tilde{\mathrm{g}} \times \tilde{\mathrm{g}})$ is an operator adjoint to the Frechet derivative $P' : T(\tilde{\mathrm{g}} \times \tilde{\mathrm{g}}) \to T(\tilde{\mathrm{g}}^* \oplus \tilde{\mathrm{g}}^*)$ of the Backlund transformation (7) and $(P'^*)^{-1}$ is an operator inverse to P', on $\tilde{\mathrm{g}}_+ \times \tilde{\mathrm{g}}_+$ and the Hamiltonians $\bar{H}_j \in \mathcal{D}(\tilde{\mathrm{g}}_+ \times \tilde{\mathrm{g}}_+)$, $j \in \mathbb{N}$:*

$$\bar{H}_j(\mathcal{A}, \mathcal{B}) := \gamma_j(l)|_{l=\mathcal{A}\mathcal{B}^{-1}} + \gamma_j(\tilde{l})|_{\tilde{l}=\mathcal{B}^{-1}(\mathcal{A}-c\partial\mathcal{B}/\partial y)}.$$

The reductions of the hierarchy (6) on the coadjoint action orbits for the central extension $\hat{\mathrm{g}}$ with taking into account the Backlund transformation (7) lead to new integrable hierarchies of nonlinear dynamical systems on matrix functional supermanifolds of two commuting and one anticommuting independent variables, being Hamiltonian ones and possessing infinite sequences of conservation laws.

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The some solution of the Bryan-Pidduck equation

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The nonlinear integrodifferential Boltzmann equation [1] that describes the evolution of rarefied gases is one of the main equations of the kinetic theory of gases. The Boltzmann equation for the model of rough spheres (or the Bryan–Pidduck equation) has the form

$$D(f) = Q(f, f); \tag{1}$$

$$D(f) \equiv \frac{\partial f}{\partial t} + \left(V, \frac{\partial f}{\partial x}\right), \tag{2}$$

$$\begin{gathered} Q(f,f) \equiv \frac{d^2}{2} \int_{R^3} dV_1 \int_{R^3} d\omega_1 \int_{\Sigma} d\alpha B(V - V_1, \alpha) \\ \Big[f(t, V_1^*, x, \omega_1^*) f(t, V^*, x, \omega^*) - f(t, V, x, \omega) f(t, V_1, x, \omega_1) \Big]. \end{gathered} \tag{3}$$

The problem of determination of the exact and approximate solutions of the Bryan–Pidduck equation in the explicit form is quite urgent. At present, the sole known exact solution of the Boltzmann equation is an expression usually called the Maxwell distribution or simply Maxwellian (after J. C. Maxwell, Scottish physicist). In the case of Maxwellians M, we get

$$D(f) = 0, \quad Q(f, f) = 0. \tag{4}$$

The solution to this equation (1)-(3) will be look for in the next form

$$f(t, x, V, \omega, u) = \int_{\mathbb{R}^3} \varphi(t, x, u) M(V, \omega, u) du. \tag{5}$$

As a measure of the deviation between the parts of equation (1) we will consider a uniform-integral error of the form:

$$\Delta = \sup_{(t,x)\in\mathbb{R}^4} \int_{\mathbb{R}^3} dV \int_{\mathbb{R}^3} d\omega \Big| D(f) - Q(f, f) \Big|. \tag{6}$$

In the paper [2], we were obtained sufficient conditions for the coefficient functions and hydrodynamic parameters appearing in the distribution, which enable one to make the analyzed error (6) as small as desired.

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Disjoint dynamical properties of wedge operators

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Let H be a separable Hilbert space and $B_0(H)$ be the C^*-algebra of compact operators on H. Given an invertible bounded operator W and a unitary operator $\mathcal{U}$ on H, we let $T_{\mathcal{U},\mathcal{W}}$ be the operator on $B_0(H)$ given by $T_{\mathcal{U},W}(F) = WF\mathcal{U}$ for all $F \in B_0(H)$. Such operators are called wedge operators. In this talk, we characterize disjoint hypercyclic finite sequences of wedge operators. We provide also sufficient conditions for a finite sequence of the dual wedge operators to be disjoint topologically transitive. Finally, we give concrete examples and applications. The talk will be based on [1].

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On semi-symmetric (α, β, γ)-inverse quasigroup

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Quasigroups and loops are generalizations of groups (see [2, 9, 10]).

Definition 1. Let $(Q, \cdot)$ be a system of non-empty set Q and a binary operation $(\cdot)$. $(Q, \cdot)$ will be called a quasigroup if for $a, b \in Q$, the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions (x, y) $\in Q \times Q$.

Definition 2. A quasigroup $(Q, \cdot)$, in which there is a unique element $\mu \in Q$, such that $x \cdot \mu = x = \mu \cdot x \ \ \forall\, x \in Q$, is called a loop. The element μ is called the identity element in Q.

In associative algebraic systems, the notion of an inverse element or property holds significance only when the system possesses a neutral element, as seen in groups, for instance. Nevertheless, in quasigroups, the inverse property can be meaningfully established even when there is no neutral element present.

Definition 3. A quasigroup $(Q, \cdot)$ will be said to have the inverse property if there are permutations on Q: $J_\lambda : x \longrightarrow x^\lambda$ and $J_\rho : x \longrightarrow x^\rho$ such that $x^\lambda(xy) = y$ and $(yx)x^\rho = y$ for $x, y \in Q$.

Certain varieties of quasigroups or loops lack the inverse property, yet exhibit characteristics that can be viewed as variations of the inverse property.

Definition 4. A quasigroup $(Q, \cdot)$ has the cross-inverse-property (and is called a CIP quasigroup) if there exists a permutation $J : \ Q \to Q;\ x \longmapsto xJ$ such that either of the following holds: $(x \cdot y) \cdot xJ = y$ or $xJ \cdot (y \cdot x) = y$ for all $x,\ y \in Q$. If $(Q, \cdot)$ is a loop with the neutral element μ, then $J = J_\lambda$ or $J = J_\rho$ and we have a CIP loop.

This class of quasigroup and loop, and their generalizations have been studied and found to be applicable to cryptography (see [7, 3]). Among such generalizations is the m-inverse quasigroup and loop (see [4]) .

Definition 5. If there is a permutation J of elements of a quasigroup $(Q, \cdot)$ such that $\forall\ x, y \in Q$ $(x \cdot y)J^m \cdot xJ^{m+1} = yJ^m$, where m is an integer, then $(Q, \cdot)$ is called an m-inverse quasigroup. In the special case $(Q, \cdot)$ is a loop with neutral element μ and $x \cdot xJ = \mu$ for all $x \in Q$, then we have an m-inverse loop.

Another of such is the (r, s, t)-inverse quasigroup (see [1, 5, 6]) which (α, β, γ)-inverse quasigroup generalizes.

Definition 6. If there is a permutation J of elements of a quasigroup $(Q, \cdot)$ such that $\forall\ x, y \in Q$ $(x \cdot y)J^r \cdot xJ^s = yJ^t$, where r, s and t are integers, then $(Q, \cdot)$ is called an (r, s, t)-inverse quasigroup. If in addition, $(Q, \cdot)$ is a loop and the permutation J is such that $x \cdot xJ = \mu$, where μ is the neutral element in Q, then $(Q, \cdot)$ is an (r, s, t)-inverse loop.

A quasigroup $(Q, \cdot)$ will be called an (α, β, γ)-inverse quasigroup, if there exist fixed permutations α, β and γ of Q, such that $(x \cdot y)\alpha \cdot x\beta = y\gamma \ \ \forall\ (x, y) \in Q \times Q$.

Conjecture 7. *A quasigroup can have more than one triple of bijections* (α, β, γ)*, for which the* (α, β, γ)*-inverse property holds.*

In this work, examples were given to illustrate that a quasigroup can have more than one (α, β, γ)-inverse property.

Definition 8. Let $(Q, \cdot)$ be a quasigroup. Define the set Δ_Q as follows:

$$\Delta_Q := \{\omega = \langle \alpha, \beta, \gamma \rangle \colon (x \cdot y)\alpha \cdot x\beta = y\gamma,\ x, y \in Q\}$$

where α, β, and γ are permutations of Q.

Definition 9. A quasigroup $(Q, \cdot)$ is said to be semi-symmetric if it satisfies the identity $(x \cdot y) \cdot x = y$ for all $x, y \in Q$.

For non-empty set Δ_Q of quasigroup $(Q, \cdot)$, it was shown that if the semi-symmetry law holds in $(Q, \cdot)$, it induces a binary operation on Δ_Q for which Δ_Q is a group.

Theorem 10. *Let* $(Q, \cdot)$ *be an* (α, β, γ)*-inverse quasigroup. If* $(Q, \cdot)$ *is semi-symmetric, then there exists a binary operation* $\otimes$ *on* Δ_Q*, such that* $(\Delta_Q, \otimes)$ *is a group.*

Conjecture 11. *There is relationship between* Δ_Q *and the autotopism group* $ATP(Q)$*, for a quasigroup* $(Q, \cdot)$*.*

Interestingly, this relation is actually an isomorphism between $\varDelta_Q$ and the autotopism group of $(Q, \cdot)$.

Theorem 12. *For an (α, β, γ)-inverse quasigroup $(Q, \cdot)$ that is semi-symmetric, $(\varDelta_Q, \otimes)$ and $ATP(Q)$ are isomorphic i.e $\varDelta_Q \cong ATP(Q)$.*

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About Rolewicz theorem on inversion of continuous bijection between F-spaces

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Well-known result of Stefan Banach states that if X and Y are F-spaces and $f : X \to Y$ is a bijective additive continuous mapping, then the inverse mapping $f^{-1} : Y \to X$ is continuous. In general case the inverse mapping can be everywhere discontinuous.

In article [1] Stefan Rolewicz presented sufficient conditions on spaces X and Y under which the inverse mapping to a continuous bijection belongs to the first Baire class.

Theorem 1 (Rolewicz, 1958). *Let X, Y be F-spaces and let X be separable locally compact. Then for every continuous bijection $f : X \to Y$ the inverse mapping $f^{-1} : Y \to X$ is Baire 1.*

The aim of this talk is a discussion of possible generalizations of the above mentioned result of Rolewicz on spaces X which are not linear. In order to do this we introduce a notion of weak Rolewicz space and prove the auxiliary fact about uniform limit of Baire 1 functions which is of self contained interest and extends corresponding results from [2].

Definition 2. A metric space (X, d) is called a *weak Rolewicz space*, if there exist $C > 0$, a sequence $(\varepsilon_n)_{n=1}^{\infty}$ of positive reals which tends to zero and a sequence $(R_n)_{n=1}^{\infty}$ of functions $R_n : X \times X \to X$ such that for all $x, y \in X$

1. if $d(x, y) \le \varepsilon_n$, then $R_n(x, y) = x$,
2. $d(R_n(x, y), y) \le C \cdot \varepsilon_n$ for $n = 1, 2, \dots$.

Every convex subset of a metric vector space is an example of a weak Rolewicz space. Moreover, there are zero dimensional examples of Rolewicz spaces.

Proposition 3. *If Y is a weak Rolewicz space, then a uniform limit $f : X \to Y$ of a sequence of Baire 1 functions $f_n : X \to Y$ belongs to the first Baire class.*

The next theorem is the main result of the talk.

Theorem 4. *Let X, Y be metric spaces and X is locally compact weak Rolewicz space. Then for every continuous bijection $f : X \to Y$ the inverse mapping $f^{-1} : Y \to X$ is Baire 1.*

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On boundary controllability problems for the heat equation with variable coefficients on a half-axis

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Consider the control system for the heat equation on a half-axis:

$$w_t = \frac{1}{\rho}\left(k w_x\right)_x + \gamma w, \qquad x \in \mathbb{R}_+,\ t \in (0, T), \tag{1}$$

$$\left(\sqrt{k/\rho}\, w_x\right)\Big|_{x=0} = u, \qquad t \in (0, T), \tag{2}$$

$$w(\cdot, 0) = w^0, \qquad x \in \mathbb{R}_+, \tag{3}$$

where $\mathbb{R}_+ = (0, +\infty)$; T is a positive constant; ρ, k, γ, w^0 are given functions; $u \in L^\infty(0, T)$ is a control. We assume ρ and k are positive on $[0, +\infty)$, $\rho, k \in C^1[0, +\infty)$, $(\rho k) \in C^2[0, +\infty)$, $(\rho k)'(0) = 0$, and

$$\sigma(x) = \int_0^x \sqrt{\rho(\mu)/k(\mu)}\, d\mu \to +\infty \qquad \text{as } x \to +\infty.$$

Moreover, we assume

$$(Q_2(\rho, k) - \gamma) \in L^\infty(0, +\infty) \bigcap C^1[0, +\infty) \qquad \text{and} \qquad \sigma\sqrt{\rho/k}\,(Q_2(\rho, k) - \gamma) \in L^1(0, +\infty),$$

where $Q_2(\rho,k)=\sqrt{k/\rho}\big(Q_1(\rho,k)\big)'+\big(Q_1(\rho,k)\big)^2$, $Q_1(\rho,k)=\sqrt{k/\rho}(k\rho)'/(4k\rho)$.

Control system (1)–(3) is considered in modified Sobolev spaces. Let $\varphi\in L^2_{\mathrm{loc}}(\mathbb{R}_+)$. We define the modified derivative $\mathbb{D}_{\rho k}$ by the rule

$$\mathbb{D}_{\rho k}\varphi=\sqrt{k/\rho}\varphi'+Q_1(\rho,k)\varphi.$$

If, in addition, $\mathbb{D}_{\rho k}\varphi\in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ and $(\mathbb{D}_{\rho k}\varphi)'\in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ we can consider $\mathbb{D}^2_{\rho k}\varphi$:

$$\mathbb{D}^2_{\rho k}\varphi=\frac{1}{\rho}\big(k\varphi'\big)'+Q_2(\rho,k)\varphi.$$

Obviously, $\mathbb{D}^m_{\rho k}\varphi=\varphi^{(m)}$ if $\rho=k=1$, $m=0,1$.

Denote

$$L^2_\rho(\mathbb{R}_+)=\{f\in L^2_{\mathrm{loc}}(\mathbb{R}_+)\mid\sqrt{\rho}f\in L^2(\mathbb{R}_+)\}$$

with the norm $\|f\|_{L^2_\rho(\mathbb{R}_+)}=\|\sqrt{\rho}f\|_{L^2(\mathbb{R}_+)}$, $f\in L^2_\rho(\mathbb{R}_+)$.

Now consider the modified Sobolev spaces

$$\mathring{\mathbb{H}}^0=L^2_\rho(\mathbb{R}_+),\quad \mathring{\mathbb{H}}^1=\{\varphi\in L^2_\rho(\mathbb{R}_+)\mid\mathbb{D}_{\rho k}\varphi\in L^2_\rho(\mathbb{R}_+)\text{ and }\varphi(0^+)=0\}$$

with the norm

$$[\![\varphi]\!]^p=\left(\sum_{m=0}^{p}\binom{p}{m}\left(\|\mathbb{D}^m_{\rho k}\varphi\|_{L^2_\rho(\mathbb{R}_+)}\right)^2\right)^{1/2},\quad \varphi\in\mathring{\mathbb{H}}^p,\quad p=0,1,$$

and the dual space $\mathring{\mathbb{H}}^{-p}=\left(\mathring{\mathbb{H}}^p\right)^*$, $p=0,1$, with the norm associated with the strong topology of the adjoint space.

In control system (1)–(3), we suppose $\left(\frac{d}{dt}\right)^p w:[0,T]\to\mathring{\mathbb{H}}^{1-2p}$, $p=0,1$; $w^0\in\mathring{\mathbb{H}}^1$.

Let $T>0$, $w^0\in\mathring{\mathbb{H}}^1$. By $\mathcal{R}_T(w^0)$, denote the set of all states $w^T\in\mathring{\mathbb{H}}^1$ for which there exists a control $u\in L^\infty(0,T)$ such that there exists a unique solution w to (1)–(3) and $w(\cdot,T)=w^T$.

Definition 1. A state $w^0\in\mathring{\mathbb{H}}^1$ is said to be *null-controllable* with respect to system (1)–(3) in a given time $T>0$ if $0\in\mathcal{R}_T(w^0)$.

Definition 2. A state $w^0\in\mathring{\mathbb{H}}^1$ is said to be *approximately controllable* to a state $w^T\in\mathring{\mathbb{H}}^1$ with respect to system (1)–(3) in a given time $T>0$ if $w^T\in\overline{\mathcal{R}_T(w^0)}$, where the closure is considered in the space $\mathring{\mathbb{H}}^1$.

Consider also the control system with the simplest heat operator (the case $\rho=k=1$, $\gamma=0$):

$$z_t=z_{yy},\qquad y\in\mathbb{R}_+,\ t\in(0,T),\tag{4}$$

$$z_y(0,\cdot)=v,\qquad t\in(0,T),\tag{5}$$

$$z(\cdot,0)=z^0,\qquad y\in\mathbb{R}_+,\tag{6}$$

where $v\in L^\infty(0,T)$ is a control, $\left(\frac{d}{dt}\right)^m z:[0,T]\to H^{1-2m}$, $m=0,1$, $z^0\in H^1$. Here H^p, $p=-1,0,1$, are the Sobolev spaces.

Controllability problems for system (4)–(6) were investigated in [1].

To study controllability problems for system (1)–(3), we use the transformation operator $\widehat{\mathbb{T}}:H^{-1}\to\mathring{\mathbb{H}}^{-1}$. It was introduced and studied in [2]. In particular, it has been proved therein that $\widehat{\mathbb{T}}$ is a continuous one-to-one mapping between the spaces H^p and $\mathring{\mathbb{H}}^p$, $p=-1,0,1$.

In the present talk, we prove that the transformation operator $\widehat{\mathbb{T}}$ is one-to-one mapping between the sets of the solutions to system (4)–(6) and to system (1)–(3). The application of the operator $\widehat{\mathbb{T}}$ allows us to conclude that the control system (1)–(3) replicates the controllability properties of the control system (4)–(6) and vice versa. A relation between controls u and v is also found. Thus, using obtained results for control system (4)–(6), we obtain the following main results for control system (1)–(3).

Theorem 3. *If a state* $w^0 \in \overset{\circ}{\mathbb{H}}{}^1$ *is null-controllable with respect to system* (1)–(3) *in a time* $T > 0$, *then* $w^0 = 0$.

Theorem 4. *Each state* $w^0 \in \overset{\circ}{\mathbb{H}}{}^1$ *is approximately controllable to any target state* $w^T \in \overset{\circ}{\mathbb{H}}{}^1$ *with respect to system* (1)–(3) *in a given time* $T > 0$.

All obtained results have been published in [3].

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On explicit reconstruction of real algebraic maps locally like moment maps

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Real algebraic geometry studies real algebraic varieties. Nash and Tognoli's theory shows that a smooth closed manifold is realized as the zero set of a real algebraic map and non-singular. It is also well-known that smooth maps between non-singular real algebraic manifolds are approximated by real algebraic maps in general. For related history and terminologies and notions, see [5] for example.

Our study is on explicit construction of real algebraic sets and maps. The k-dimensional unit sphere S^k in the $(k+1)$-dimensional real affine space $\mathbb{R}^{k+1}$ and its canonical embedding into higher dimensional real affine spaces (and the compositions with canonical projections) give simplest examples. In general, it is very difficult to give real algebraic sets and important real algebraic maps on them very explicitly. Here, we reconstruct real algebraic maps generalizing the canonical projections of the unit spheres $S^k \subset \mathbb{R}^{k+1}$ from given regions in the target spaces. We discuss this from the viewpoint of global singularity theory and differential topology of manifolds.

Theorem 1. [2] *Let* l_1, l_2 *and* n *be positive integers. Let* $D \subset \mathbb{R}^n$ *be an open subset. Let* $\{S_j\}_{j=1}^{l_1}$ *be a family of non-singular real algebraic hypersurfaces of* $\mathbb{R}^n$. *Let* S_j *be also the zero set of a real polynomial* f_j. *We also assume the following.*

(1) *It holds that* $D \bigcap \bigcup_{j=1}^{l_1} S_j = \emptyset$, *that* $\overline{D} \bigcap S_j \neq \emptyset$ *for* $1 \leq j \leq l_1$ *and that* $\overline{D} - D \subset \bigcup_{j=1}^{l_1} S_j$. *For any small open neighborhood* U_D *of* $\overline{D}$, $D = U_D \bigcap \bigcap_{j=1}^{l_1} \{x \mid f_j(x) > 0\}$ *and* $\overline{D} = U_D \bigcap \bigcap_{j=1}^{l_1} \{x \mid f_j(x) \geq 0\}$.

(2) *Let* $\{i_j\}_{j=1}^{i_0}$ *be an increasing sequence of integers such that* $1 \leq i_j \leq l_1$. *Let* $p \in \bigcap_{j=1}^{i_0} S_{i_j} \bigcap \overline{D}$ *and for any increasing sequence* $\{{i_j}'\}_{j=1}^{i_0+1}$ *of integers satisfying* $1 \leq {i_j}' \leq l_1$ *and containing* $\{i_j\}_{j=1}^{i_0}$ *as a subsequence,* $p \notin \bigcap_{j=1}^{i_0+1} S_{{i_j}'}$ *hold. Let* $e_j : S_j \rightarrow \mathbb{R}^n$ *denote the canonical embedding. Assume that* $\bigcap_{j=1}^{i_0} S_{i_j}$ *is an* $(n - i_0)$*-dimensional smooth submanifold of* $\mathbb{R}^n$ *with no boundary and let* $e_{\{i_j\}_{j=1}^{i_0}} : \bigcap_{j=1}^{i_0} S_{i_j} \rightarrow \mathbb{R}^n$ *denote the canonical embedding. Then the intersection* $\bigcap_{j=1}^{i_0} de_j(T_p S_j)$ *and the image* $de_{\{i_j\}_{j=1}^{i_0}}(T_p(\bigcap_{j=1}^{i_0} S_{i_j}))$ *of the differential of* $e_{\{i_j\}_{j=1}^{i_0}}$ *at* p *always agree.*

(3) *There exists a map* m_{l_1,l_2} *which maps each integer* $1 \leq i \leq l_1$ *to an integer* $1 \leq i' \leq l_2$, *which is a surjection to the set of all integers* $1 \leq j \leq l_2$, *and whose restriction to an increasing sequence* $\{i_j\}_{j=1}^{i_0}$ *making* $\bigcap_{j=1}^{i_0} S_{i_j} \bigcap \overline{D}$ *a non-empty set is always injective.*

Let m_{l_2} *be a map mapping each integer* $1 \leq i \leq l_2$ *to a non-negative integer. Let* $m := n + \Sigma_{j=1}^{l_2} m_{l_2}(j)$. *Then there exist an* m*-dimensional non-singular real algebraic manifold* $M \subset \mathbb{R}^{m+l_2}$ *and a real algebraic map* $f : M \rightarrow \mathbb{R}^n$ *such that* $f(M) = \overline{D}$, *that the image of the singular set is* $\overline{D} - D$, *and that each preimage* $f^{-1}(p)$ $(p \in \overline{D})$ *is a single-point set or a product of spheres and at most* $(m - n)$*-dimensional.*

We present the case where hypersurfaces S_j do not intersect and $l_2 = 1$. Here $M := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{m-n+1} = \mathbb{R}^{m+1} \mid \prod_{j=1}^{l_1}(f_j(x)) - {||y||}^2 = 0\}$. The map $f : M \rightarrow \mathbb{R}^n$ is the composition of the canonical embedding into $\mathbb{R}^{m+1}$ with the canonical projection. In the case $\overline{D} := D^n$, the n-dimensional unit sphere, we have the canonical projection of the unit sphere $S^m \subset \mathbb{R}^{m+1}$ into $\mathbb{R}^n$. This is already done in [1, 3]. In the case the hypersurfaces S_j do not intersect, we have so-called *special generic* maps presented in [4]. See also [6] for special generic maps in the theory of differential topology of manifolds. Our result reconstructs real algebraic maps locally like moment maps.

As an application, Theorem 2 presents explicit families of real algebraic functions which are compositions of maps into $\mathbb{R}^2$ obtained through Theorem 1 with the canonical projection ([2]). This is seen as extending some result of [1]: we have studied cases where the hypersurfaces S_j do not intersect. We define the *Reeb graph* W_c of a smooth function c on a closed manifold X. Let $\sim_c$ be the following equivalence relation on X: $x_1 \sim_c x_2$ if and only if x_1 and x_2 are in a same connected component of a preimage $c^{-1}(y)$ $(y \in \mathbb{R})$. Let $q_c : X \rightarrow W_c := X/\sim_c$ denote the quotient map. The vertex set is the set of all points v such that ${q_c}^{-1}(v)$ contain some critical points of c. See also [7] for example.

Theorem 2. [2] *In Theorem 1, let* $n = 2$, $l_1 := l_{0,1}$, $\{S_j := S_{0,j}\}_{j=1}^{l_1}$ *consist of mutually disjoint circles and* $D := D_0 \subset \mathbb{R}^2$ *be a connected and bounded open set surrounded by the disjoint union* $\sqcup_{j=1}^{l_{0,1}} S_j$.

We construct a map $f := f_{M_0} : M := M_0 \rightarrow \mathbb{R}^2$ *by Theorem 1. We consider the function* $f_{0,M_0} := \pi_{2,1} \circ f_{M_0}$ *where* $\pi_{2,1} : \mathbb{R}^2 \rightarrow \mathbb{R}$ *is the canonical projection. Let* $m \geq 4$ *be an integer. For each non-negative integer* k, *we can consider some situation enjoying the following and apply Theorem 1.*

(1) $n = 2$. $l_1 := l_{0,1} + k$. $l_2 := 2$.

(2) *The family* $\{S_j\}_{j=1}^{l_1}$ *is a family of circles enjoying the following.*

   (a) *For* $1 \leq j \leq l_{0,1}$, $S_j := S_{0,j}$ *and the family* $\{S_{l_{0,1}+j}\}_{j=1}^{k}$ *consists of mutually disjoint and sufficiently small circles in* $\mathbb{R}^2$ *centered at points in some circles in* $\{S_j := S_{0,j}\}_{j=1}^{l_1}$.
   (b) *For each* $S_{l_{0,1}+j_2}$ $(1 \leq j_2 \leq k)$, *there exists a unique circle* S_{0,j_1} *such that* $S_{l_{0,1}+j_2} \bigcap S_{0,j_1} \neq \emptyset$. *Furthermore, the non-empty intersection is a two-point set.*

(3) The open set $D \subset \mathbb{R}^2$ *is the connected and bounded component of the complementary set of* $\bigcup_{j=1}^{l_1} S_j$ *in* $\mathbb{R}^2$ *which is also a subset of* D_0.

Furthermore, choosing suitable situations, we have a family $\{f := f_{M_k} : M := M_k \to \mathbb{R}^2\}_{k=0}^{\infty}$ *of real algebraic maps on* m*-dimensional closed and connected manifolds enjoying the following properties.*

(4) The Reeb graph $W_{f_{0,M_k}}$ *of the function* $f_{0,M_k} := \pi_{2,1} \circ f_{M_k}$ *collapses to the Reeb graph* $W_{f_{0,M_0}}$.
(5) The graphs $W_{f_{0,M_{k_1}}}$ *and* $W_{f_{0,M_{k_2}}}$ *are not isomorphic for distinct integers* k_1 *and* k_2.

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Group classification of Kolmogorov backward equations with power diffusivity

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We carry out the complete group classification of the class $\mathcal{F}$ of Kolmogorov backward equations with power diffusivity

$$\mathcal{F}_{\alpha\beta}\colon\ u_t + xu_y = |x-\alpha|^\beta u_{xx},$$

where α and β are arbitrary real parameters, by solving the same problem for the class $\mathcal{F}'$ of the equations of the form

$$\mathcal{F}'_\beta\colon\quad u_t + xu_y = |x|^\beta u_{xx},$$

with β remains to be the only arbitrary element in the class $\mathcal{F}'$. Using the modified version of the direct method, we compute the equivalence groupoids $\mathcal{G}^\sim_{\mathcal{F}}$ and $\mathcal{G}^\sim_{\mathcal{F}'}$ of the classes $\mathcal{F}$ and $\mathcal{F}'$, respectively, and consequently show that the class $\mathcal{F}'$ is semi-normalized in the usual sense. The modification of the direct method is based on embedding both the classes $\mathcal{F}$ and $\mathcal{F}'$ into the class $\bar{\mathcal{F}}$ of ultraparabolic (1+2)-dimensional Fokker–Planck equations of the form

$$u_t + B(t,x,y)u_y = A^2(t,x,y)u_{xx} + A^1(t,x,y)u_x + A^0(t,x,y)u + C(t,x,y),$$

where the tuple $\bar\theta := (B, A^2, A^1, A^0, C)$ of arbitrary elements of the class $\bar{\mathcal{F}}$ runs through the solution set of the system of the inequalities $A^2 \neq 0$ and $B_x \neq 0$ with no restrictions on A^0, A^1 and C. The equivalence groupoid of of the class $\bar{\mathcal{F}}$ was described in [1, 2] via presenting the equivalence group of this class and stating that it is normalized, see [3] for required notions, results and further references. We use the known determining equations for admissible transformations within the superclass $\bar{\mathcal{F}}$ as the known principal constraints for admissible transformations within the classes $\mathcal{F}$ and $\mathcal{F}'$. After explicitly constructing the groupoids $\mathcal{G}^\sim_{\mathcal{F}}$ and $\mathcal{G}^\sim_{\mathcal{F}'}$, it is easy to show that the group classification of the class $\mathcal{F}$ reduces to that of the class $\mathcal{F}'$.

The class $\mathcal{F}'$ admits a distinguished discrete equivalence transformation

$$\mathscr{J}\colon\quad \tilde t = y\operatorname{sgn} x,\quad \tilde x = \frac{1}{x},\quad \tilde y = t\operatorname{sgn} x,\quad \tilde u = \frac{u}{x},\quad \tilde\beta = 5-\beta,$$

which turns out to be the only point equivalence transformation essential for carrying out the group classification of this class modulo the $\mathcal{G}^\sim_{\mathcal{F}'}$-equivalence.

The following chain of assertions provides the complete solutions to the group classification problems for the classes $\mathcal{F}$ and $\mathcal{F}'$.

Theorem 1. (i) *The point transformations* $\mathscr{S}(c_1)$: $(\tilde t, \tilde x, \tilde y, \tilde u, \tilde\beta, \tilde\alpha) = (t, x+c_1, y+c_1t, u, \beta, \alpha+c_1)$ *where* c_1 *is arbitrary constant, constitute a one-parameter group of equivalence transformations of the class* $\mathcal{F}$.

(ii) *The wide family of admissible transformations* $\mathcal{S}_{\alpha\beta} := \big((\alpha,\beta), \pi_*\mathscr{S}(-\alpha), (0,\beta)\big)$ *of the class* $\mathcal F$ *from the action groupoid of its equivalence group maps this class onto the class* $\mathcal F'$ *interpreted as a subclass of* $\mathcal F$.

(iii) *The point transformation* $\mathscr{J}$ *is a (discrete) equivalence transformation of the class* $\mathcal F'$.

(iv) *The class* $\mathcal F'$ *is semi-normalized with respect to the discrete equivalence subgroup generated by* $\mathscr{J}$. *In other words, the equivalence groupoid* $\mathcal{G}^{\sim}_{\mathcal F'}$ *of* $\mathcal F'$ *is the Frobenius product of the action groupoid of this subgroup and the fundamental equivalence groupoid* $\mathcal{G}^{\rm f}_{\mathcal F'}$ *of* $\mathcal F'$.

Corollary 2. (i) *Different equations* $\mathcal F'_\beta$ *and* $\mathcal F'_{\tilde\beta}$ *are similar with respect to point transformations if and only if* $\beta+\tilde\beta=5$.

(ii) *Equations* $\mathcal F_{\alpha\beta}$ *and* $\mathcal F_{\tilde\alpha\tilde\beta}$ *are similar with respect to point transformations if and only if either* $\tilde\beta=\beta$ *or* $\beta+\tilde\beta=5$.

(iii) *The equivalence groupoid* $\mathcal{G}^{\sim}_{\mathcal F}$ *of* $\mathcal F$ *is generated by admissible transformations* $\mathcal{S}_{\alpha\beta}$ *and elements of* $\mathcal{G}^{\sim}_{\mathcal F'}$. *More specifically, for each admissible transformation* $\big((\alpha,\beta),\Phi,(\tilde\alpha,\tilde\beta)\big)$ *of* $\mathcal F$, *we have* $\Phi=\pi_*\mathscr{S}(\tilde\alpha)\circ\breve\Phi\circ\pi_*\mathscr{S}(-\alpha)$ *for some point transformation* $\breve\Phi$ *with* $(\beta,\breve\Phi,\tilde\beta)\in\mathcal{G}^{\sim}_{\mathcal F'}$.

Theorem 3. *The kernel Lie invariance algebra* $\mathfrak g^{\cap}_{\mathcal F'}$ *of the equations from the class* $\mathcal F'$ *is*

$$\mathfrak g^{\cap}_{\mathcal F'}=\langle\mathcal P^t,\mathcal P^y,\mathcal I,(tx-y)\partial_u,x\partial_u,\partial_u\rangle,\quad \textit{where}\quad \mathcal P^t:=\partial_t,\quad \mathcal P^y:=\partial_y,\quad \mathcal I:=u\partial_u.$$

Any equation $\mathcal F'_\beta$ *from* $\mathcal F'$ *is invariant with respect to the algebra*

$$\mathfrak g^{\rm gen}_\beta=\langle\mathcal P^t,\mathcal P^y,\mathcal I,\mathcal D^\beta,\mathcal Z(f^\beta)\rangle\ \ \textit{with}\ \ \mathcal D^\beta:=(2-\beta)t\partial_t+x\partial_x+(3-\beta)y\partial_y,\ \mathcal Z(f^\beta):=f^\beta\partial_u,$$

where the parameter function $f^\beta=f^\beta(t,x,y)$ *runs through the solution set of this equation, and* $\beta\in(-\infty,5/2]$ *modulo the* $G^{\sim}_{\mathcal F'}$*-equivalence. the maximal Lie invariance algebra* $\mathfrak g_\beta$ *of the equation* $\mathcal F_\beta$ *coincides with* $\mathfrak g^{\rm gen}_\beta$ *if and only if* $\beta\in\mathbb R\setminus\{0,2,3,5\}$. *A complete list of* $G^{\sim}_{\mathcal F'}$*-inequivalent essential Lie symmetry extensions in the class* $\mathcal F'$ *is exhausted by the following cases:*

$$\beta=2\colon\quad \mathfrak g_2=\mathfrak g^{\rm gen}_2\dotplus\langle\mathcal K_2\rangle\quad\textit{with}\quad \mathcal K_2=2xy\partial_x+y^2\partial_y-xu\partial_u,$$

$$\begin{aligned}\beta=0\colon\quad &\mathfrak g_0=\mathfrak g^{\rm gen}_0\dotplus\langle\mathcal K_0,\mathcal P^3,\mathcal P^2,\mathcal P^1\rangle\quad\textit{with}\\ &\mathcal K_0=t^2\partial_t+(tx+3y)\partial_x+3ty\partial_y-(x^2+2t)u\partial_u,\\ &\mathcal P^3=3t^2\partial_x+t^3\partial_y+3(y-tx)u\partial_u,\quad \mathcal P^2=2t\partial_x+t^2\partial_y-xu\partial_u,\quad \mathcal P^1=\partial_x+t\partial_y.\end{aligned}$$

Corollary 4. *The kernel Lie invariance algebra* $\mathfrak g^{\cap}_{\mathcal F}$ *of the equations from the class* $\mathcal F$ *coincides with that for the class* $\mathcal F'$, $\mathfrak g^{\cap}_{\mathcal F}=\mathfrak g^{\cap}_{\mathcal F'}$ *Any equation* $\mathcal F_{\alpha\beta}$ *from* $\mathcal F$ *is invariant with respect to the algebra*

$$\mathfrak g^{\rm gen}_{\alpha\beta}=\big\langle\mathcal P^t,\mathcal P^y,\mathcal I,\mathcal D^{\alpha\beta},\mathcal Z(f^{\alpha\beta})\big\rangle$$

with $\mathcal D^{\alpha\beta}:=(2-\beta)t\partial_t+(x-\alpha)\partial_x+\big((3-\beta)y-\alpha t\big)\partial_y$, $\mathcal Z(f^{\alpha\beta}):=f^{\alpha\beta}\partial_u$, *and the parameter function* $f^{\alpha\beta}=f^{\alpha\beta}(t,x,y)$ *running through the solution set of this equation. Modulo the* $\mathcal{G}^{\sim}_{\mathcal F'}$*-equivalence, we can assume* $\beta\in(-\infty,5/2]$, *and a complete list of* $\mathcal{G}^{\sim}_{\mathcal F'}$*-inequivalent essential Lie symmetry extensions in the class* $\mathcal F$ *is exhausted by the counterparts of those in the class* $\mathcal F'$, $\mathcal F_{00}$ *and* $\mathcal F_{02}$. *An analogous list up to the* $G^{\sim}_{\mathcal F'}$*-equivalence consists of the equations* $\mathcal F_{00}$, $\mathcal F_{02}$, $\mathcal F_{03}$ *and* $\mathcal F_{05}$.

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Integral problem for system of partial differential equations of third order

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Let $H(\mathbb{R}_+ \times \mathbb{R}^n)$ be a class of entire functions on $\mathbb{R}$, K_L is a class of quasipolynomials of the form $\varphi(x) = \sum_{i=1}^{n} Q_r(x) \exp[\alpha_r x]$, where $\alpha_r \in L \subseteq \mathbb{C}$, $\alpha_k \neq \alpha_l$, for $k \neq l$, $Q_r(x)$ are given polynomials.

Each quasipolynomial defines a differential operator $\varphi\left(\frac{\partial}{\partial\lambda}\right)$ of finite order on the class of entire function, in the form $\sum_{r=1}^{m} Q_r\left(\frac{\partial}{\partial\lambda}\right) \exp\left[\alpha_i \frac{\partial}{\partial\lambda}\right]\Big|_{\lambda=0}$.

In the strip $\Omega = \{(t,x) \in \mathbb{R}^{n+1} : t \in \{([T_1,T_2] \cup [T_3,T_4]), x \in \mathbb{R}^n\}$, we consider of the system of equations

$$\frac{\partial^3 U_i}{\partial t^3} + \sum_{j=1}^{n} \left\{ a_{ij}\left(\frac{\partial}{\partial x}\right) \frac{\partial^2 U_j}{\partial t^2} + b_{ij}\left(\frac{\partial}{\partial x}\right) \frac{\partial U_j}{\partial t} + c_{ij}\left(\frac{\partial}{\partial x}\right) \right\} U_j(t,x) = 0, \tag{1}$$

$$\int_{T_1}^{T_2} U_{ik}(t,x)dt + \int_{T_3}^{T_4} U_{ik}(t,x)dt = \varphi_{ik}(x), \quad k = 1,2,3, \tag{2}$$

$$\int_{T_1}^{T_2} tU_{ik}(t,x)dt + \int_{T_3}^{T_4} tU_{ik}(t,x)dt = \varphi_{ik}(x). \quad i = 1,...,n, \tag{3}$$

$$\int_{T_1}^{T_2} t^2 U_{ik}(t,x)dt + \int_{T_3}^{T_4} t^2 U_{ik}(t,x)dt = \varphi_{ik}(x). \tag{4}$$

Where $a_{ij}\left(\frac{\partial}{\partial x}\right), b_{ij}\left(\frac{\partial}{\partial x}\right), c_{ij}\left(\frac{\partial}{\partial x}\right)$, are differential expression with entire symbols $a_{ij}(\lambda) \neq 0, b_{ij}(\lambda) \neq 0, c_{ij}(\lambda) \neq 0$.

Let be $\eta(\lambda) = \int_{T_1}^{T_2} W^{n-1}(t,\lambda)dt + \int_{T_3}^{T_4} W^{n-1}(t,\lambda)dt$ is a certain function $W(t,\lambda)$ is a solution of equation $\left(\frac{d^n}{dt^n} + \sum_{i=1}^{n} a_i(\lambda) \frac{d^{n-i}}{dt^{n-i}}\right) W(t,\lambda) = 0$, satisfies conditions $W^n(t,\lambda)\Big|_{t=0} = 1, \quad W^{n-1}(t,\lambda)\Big|_{t=0} = 0, \quad W(t,\lambda)\Big|_{t=0} = 0$.

Denote be $P = \Big\{\Delta(\lambda) = 0, \lambda \in \mathbb{C}\Big\}$ set zeros of function $\eta(\lambda)$.

Theorem 1. ***Theorem.*** *Let $\varphi_{ik}(x) \in K_L$, $i = 1,...,n$, $j = 1,...,n$ then the class $K_{L \setminus P}$ exist and unique solution of the problem (1)-(4). Solution of the problem (1)-(4) can be represented in the form*

$$U_i(t,x) = \sum_{k=1}^{3} \sum_{p=1}^{n} \varphi_{kp}\left(\frac{\partial}{\partial x}\right) \left\{ \frac{1}{\eta(\lambda)} T_{kjp}(t,\lambda) W(t,\lambda) \exp[\lambda x] \right\} \Bigg|_{\lambda=0},$$

where $T_{kjp}(t,\lambda) = l^T\left(\frac{d}{dt},\lambda\right)$ is transpose of a matrix $\left(\frac{d}{dt},\lambda\right)$.

Solution of the problem (1) - (4) according to the differential-symbol method [1], [2] exists and unique in the class of quasi-polynomials. Be means of the differential-symbol method [1], [2] we construct of the problem (1)-(4). This problem is a continuos works [3] - [6].

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The density of Borromean primes

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My talk is concerned with *Arithmetic Topology*, which investigates the interactions between number theory and 3-dimensional topology. The systematic study of this subject was started by B. Mazur, M. Morishita, M. Kapranov and A. Reznikov etc. As one of the analogies in arithmetic topology, the Legendre symbol can be interpreted as Gauss's linking number ([1, Chapter 4]). In [2], Rédei attempted to generalize Gauss's genus theory and introduced a certain triple symbol $[p_1, p_2, p_3]$ for certain primes $p_1,\ p_2, p_3 \equiv 1 \bmod 4$. This symbol may be regarded as a triple generalization of the Legendre symbol $(\frac{p_1}{p_2})$, and it describes the decomposition law of p_3 in a certain dihedral extension over $\mathbb{Q}$ of degree 8, which is determined by $p_1,\ p_2$. Morishita interpreted the Rédei symbol as an arithmetic analogue of Milnor's triple linking number ([1, Chapter 9]). Now *Borromean primes* in the title is defined as arithmetic analogues of Borromean rings:

Definition 1. The triple of primes $\{p_1, p_2, p_3\}$ is called *Borromean primes* when it satisfies the following conditions:

$$p_i \equiv 1 \bmod 4\ (i = 1, 2, 3), \quad \left(\frac{p_i}{p_j}\right) = 1\ (1 \leq i \neq j \leq 3) \quad \text{and} \quad [p_1, p_2, p_3] = -1.$$

The study of asymptotic distribution of primes goes back to Gauss, and it is viewed as an origin of the so called arithmetic statistics nowadays. Gauss predicted Prime Number Theorem and it

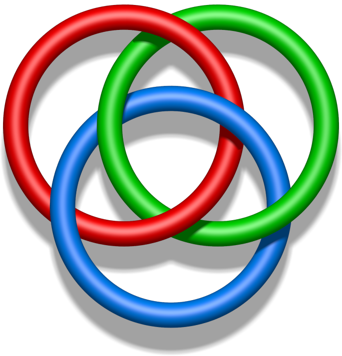

FIGURE 1.1. Borromean rings

was proved independently by Hadamard and de la Vallée Poussin. In 19th century, the following formula was shown by Dirichlet: for coprime integers $m(>1)$ and a,

$$\pi(x;a,m) \sim \frac{1}{\varphi(m)} \cdot \frac{x}{\log x},$$

where $\pi(x;a,m)$ stands for the number of primes less than or equal to x which have the form $a+km$, and $\varphi(m)$ is the Euler function. It was generalized to the following more general density theorem, known as the Chebotarev density theorem: Let M be a number field and M' be a finite Galois extension of M. For $\sigma \in \mathrm{Gal}(M'/M)$ and any positive real number x, we define

$$\pi_{M'/M}(x;\sigma) := \# \left\{ \mathfrak{p} \in S_M^0 \;\middle|\; \mathfrak{p}\text{: unramified in } M',\ \mathrm{N}_M\mathfrak{p} \le x,\ \left[\frac{M'/M}{\mathfrak{p}}\right] = C(\sigma) \right\},$$

where S_M^0, $\mathrm{N}_M\mathfrak{p}$, $[\frac{M'/M}{\mathfrak{p}}]$, and $C(\sigma)$ mean the set of prime ideals of M, the absolute norm of $\mathfrak{p} \in S_M^0$, the Artin symbol for $\mathfrak{p} \in S_M^0$ and the conjugacy class of σ in $\mathrm{Gal}(M'/M)$, respectively. Then the Chebotarev density theorem asserts

$$\pi_{M'/M}(x;\sigma) \sim \frac{\#C(\sigma)}{\#G} \cdot \frac{x}{\log x}.$$

Note that Dirichlet's theorem is interpreted as a special case of Chebotarev's theorem in the m-th cyclotomic field $\mathbb{Q}(\zeta_m)$.

In this talk, I will show the density of Borromean primes. Let $\pi_{\mathrm{Borr}}(x)$ denote the number of Borromean primes $\{p_1,p_2,p_3\}$ with $p_i \le x$ for $i=1,2,3$. Then our main theorem is stated as follows:

Theorem 2 (Main Theorem [3]). *If GRH is true, we have*

$$\lim_{x\to+\infty} \frac{\pi_{\mathrm{Borr}}(x)}{\#\{\{p_1,p_2,p_3\} \,|\, p_i \le x \text{ (for } i=1,2,3),\ p_i \ne p_j\ (i \ne j)\}} = \frac{1}{128},$$

where GRH means Generalized Riemann Hypothesis.

Note that our theorem is not obtained by a straightforward application of Chebotarev density theorems. We need for the proof more elaborate analysis on the error term of the Chebotarev density theorem under GRH ([4]).

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Invariant transformation of generalized-recurrent-parabolic spaces that are in a quasi-geodesic mapping

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We study diffeomorphisms of pseudo-Riemannian spaces that belong to the intersection of classes of quasi-geodesic mappings (QGM) [2] with the reciprocity condition and almost-geodesic mappings of the second type [1]. We mean that QGM $f : (V_n, g_{ij}, F_i^h) \longrightarrow (\overline{V}_n, \overline{g}_{ij}, F_i^h)$ satisfies the reciprocity condition if the reverse mapping f^{-1} is also QGM.

The fundamental equations of such a mapping f in the common coordinate system (x^i) with respect to the mapping f has the form:

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \psi_{(i}(x)\delta^h_{j)} + \phi_{(i}(x)F^h_{j)}(x), \tag{1}$$

$$F^h_i(x) = \overline{F}^h_i(x),$$

$$g_{i\alpha}F^\alpha_j = -g_{j\alpha}F^\alpha_i, \qquad \overline{g}_{i\alpha}F^\alpha_j = -\overline{g}_{j\alpha}F^\alpha_i, \tag{2}$$

$$F^h_{(i,j)} = q_{(i}F^h_{j)}, \tag{3}$$

$$F^h_\alpha F^\alpha_i = e\delta^h_i, \qquad e = 0, \pm 1, \tag{4}$$

$$i, h, j, ... = 1, 2, ...n,$$

where $\Gamma^h_{ij}, \overline{\Gamma}^h_{ij}$ are the Christoffel symbols of $V_n, \overline{V}_n$, respectively; $\psi_i(x)$, $\phi_i(x)$, $q_i(x)$, $p_i(x)$ are certain covectors; $F^h_i(x)$ is affinor; brackets (i, j) denote the symmetrization with respect to the corresponding indices; comma «,» is a sign of the covariant derivative in respect to the connection of V_n.

We call an affinor structure F^h_i that satisfies conditions (3) a *generalized-recurrent structure* (of elliptic, hyperbolic, or parabolic type). Let us study the case of a parabolic structure ($e = 0$).

The following holds:

Theorem 1. *If there is a non-trivial QGM of generalized-recurrent-parabolic spaces* $f : (V_n, g_{ij}, F_i^h) \longrightarrow (\overline{V}_n, \overline{g}_{ij}, F_i^h)$*, which corresponds to the affinor* F_i^h *and the vector* ϕ_i*, then it generates another non-trivial QGM of other generalized-recurrent-parabolic spaces*

$$f_1 : (\overset{1}{V}_n, \overset{1}{g}_{ij}, \overset{1}{F}{}_i^h) \longrightarrow (\overset{1}{\overline{V}}_n, \overset{1}{\overline{g}}_{ij}, \overset{1}{F}{}_i^h),$$

which corresponds to the affinor $\overset{1}{F}{}_i^h$ *and the vector* $\overset{1}{\phi}_i$ *and preserves the generalized recurrence vector* q_i*. The tensors* $\overset{1}{g}_{ij}$*,* $\overset{1}{\overline{g}}_{ij}$*,* $\overset{1}{\phi}_i$*,* $\overset{1}{F}{}_i^h$ *are given by the formulas*

$$\overset{1}{g}_{ij} = e^{2\psi} \overline{g}^{\alpha\beta} g_{\alpha i} g_{\beta j},$$

$$\overset{1}{\overline{g}}_{ij} = e^{2\psi} g_{ij},$$

$$\overset{1}{g}_{ij,k} = -\overset{1}{\phi}_{\bar{i}} g_{jk} - \overset{1}{\phi}_{\bar{j}} g_{ik} - \overset{1}{\phi}_i F_{jk} - \overset{1}{\phi}_j F_{ik},$$

$$\overset{1}{F}{}_i^h \overset{def}{=} F_\alpha^h \tilde{B}_i^\alpha = F_i^\alpha \tilde{B}_\alpha^h,$$

$$B_i^h = e^{2\psi} \overline{g}^{h\alpha} g_{\alpha i}, \qquad \tilde{B}_i^h = e^{-2\psi} g^{h\alpha} \overline{g}_{\alpha i}.$$

Theorem 2. *If there is a non-trivial QGM of generalized-recurrent-parabolic spaces* $f : (V_n, g_{ij}, F_i^h) \longrightarrow (\overline{V}_n, \overline{g}_{ij}, F_i^h)$*, which corresponds to the affinor* F_i^h *and the vector* ϕ_i*, then it generates an infinite sequence of non-trivial QGM of other generalized-recurrent-parabolic spaces*

$$\begin{array}{c}
(\overset{1}{V}_n, \overset{1}{g}_{ij}, \overset{1}{F}{}_i^h) \xrightarrow{f_1} (\overset{1}{\overline{V}}_n, \overset{1}{\overline{g}}_{ij}, \overset{1}{F}{}_i^h), \\
\downarrow \\
(\overset{2}{V}_n, \overset{2}{g}_{ij}, \overset{2}{F}{}_i^h) \xrightarrow{f_2} (\overset{2}{\overline{V}}_n, \overset{2}{\overline{g}}_{ij}, \overset{2}{F}{}_i^h), \\
\downarrow \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\downarrow \\
(\overset{s}{V}_n, \overset{s}{g}_{ij}, \overset{s}{F}{}_i^h) \xrightarrow{f_s} (\overset{s}{\overline{V}}_n, \overset{s}{\overline{g}}_{ij}, \overset{s}{F}{}_i^h), \\
\downarrow \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}$$

which correspond to the affinor $\overset{s}{F}{}_i^h$ *and the vector* $\overset{s}{\phi}_i$ *and preserve the generalized recurrence vector* q_i*. The tensors* $\overset{s}{g}_{ij}$*,* $\overset{s}{\overline{g}}_{ij}$*,* $\overset{s}{\phi}_i$*,* $\overset{s}{F}{}_i^h$*, are given by the formulas*

$$\overset{s}{g}_{ij} = \overset{(s)}{B}{}_i^\alpha g_{\alpha j}, \qquad \overset{s}{\overline{g}}_{ij} = e^{2\psi} \overset{(s-1)}{B}{}_i^\alpha g_{\alpha j}.$$

$$\overset{s}{\phi}_i = \phi_\alpha \overset{(s)}{B}{}_i^\alpha, \qquad \overset{s}{F}{}_i^h = F_\alpha^h \overset{(s)}{\tilde{B}}{}_i^\alpha,$$

where $\overset{(s)}{B}{}_i^h$ *is the s-th degree of the affinor* B_i^h *and we mean* $\overset{(0)}{B}{}_i^h = \delta_i^h$*.*

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Notes on the Quality of Non-compactness for Non-compact Sobolev Embeddings

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It is well known that when a Sobolev space on a bounded domain is embedded into the smallest possible Lebesgue or Lorentz space, the resulting embedding is non-compact. In this talk, we will closely examine non-compact Sobolev embeddings and describe the quality of their non-compactness from different points of view.

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Ordinary linear differential operators and connections. Application to curvilinear webs

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The framework is real analytic or holomorphic, the field $\mathbb{K}$ denoting $\mathbb{R}$ or $\mathbb{C}$ according to this framework.

We are given two vector bundles E and F, respectively of rank p and q, over a n-dimensional manifold V. We assume $q < p(n+1)$ (the rank of F is smaller than the rank of J^1E). A linear homogeneous differential operator of order one[1] is a linear morphism of vector bundles $D\colon J^1E \to F$. Associated to it is the partial order equation

$$(*) \quad \mathcal{D}s = 0, \text{ where we set, for any section } s \text{ of } E,\ \mathcal{D}s := D(j^1 s).$$

---

[1] The theory works also for differential operators of higher order.

The restriction $\sigma_1(D) : T^*(V) \otimes E \to F$ of D to the kernel $T^*(V) \otimes E$ of the projection $J^1E \to E$ is called the principal symbol of D.

More generally, for any integer h (≥ 1), we define by successive derivations the $(h-1)^{th}$ prolongation $D_h : J^hE \to J^{h-1}F$ of D : the solutions of $(*)$ are still the sections s of E such that $D_h(j^h s) = 0$. Recalling, the exact sequence

$$0 \to S^hT^*(V) \otimes E \to J^hE \to J^{h-1}E \to 0,$$

where $S^hT^*(V)$ denotes the h^{th} symetric power of the bundle $T^*(V)$ of 1-forms, we call *principal symbol* of D_h the restriction

$$\sigma_h(D) : S^hT^*(V) \otimes E \to J^{h-1}F$$

of D_h to the sub-bundle $S^hT^*(V) \otimes E$ of J^hE.

We denote by

$$c(n,h) := \frac{(n-1+h)!}{(n-1)!\ h!}$$

the dimension of the $\mathbb{K}$-vector space of homogeneous polynomials of degree h with n unknowns and coefficients in $\mathbb{K}$, which is also the rank of $S^hT^*(V)$, or the number of multi-indices $I = (i_1, ..., i_n)$ of partial derivatives of order $|I| = h$ with respect to n unknowns, (where $|I| = i_1 + i_2 + ... + i_n$).

Definition 1. The differential operator D is said to be ordinary if $q \leq pn$ and, for any h $(h \geq 1)$, the principal symbol $\sigma_h(D)$ has maximal rank $(inf(q.c(n,h-1), p.c(n,h)))$

If D is ordinary, the kernel R_h of D_h is a vector bundle, which is the set of the formal solutions of $(*)$ at order h $\big($with $R_{h+1} = J^1R_h \cap J^{h+1}E$, the intersection being in $J^1(J^hE)\big)$.

Definition 2. The differential operator D is said to be calibrated if $\frac{p(n-1)}{q-p}$ is an integer, strictly positive.

This implies in particular $p < q$.

If $p < q$, we have only finitely many conditions to check for D to be ordinary. In fact, set :

$$h_0 := \Big[\frac{p(n-1)}{q-p}\Big], \quad \text{(the integral part of } \tfrac{p(n-1)}{q-p}\text{)}.$$

We then get:

Proposition 3. *For D to be ordinary when $p < q \leq p.n$, it is sufficient that the principal symbols $\sigma_h(D)$ have their rank maximal for $1 \leq h \leq h_0 + 1$ in general (resp. for $1 \leq h \leq h_0$ if D is moreover calibrated).*

We first prove:

Theorem 4. *If D is ordinary and $p < q \leq p.n$, the dimension of the vector space $\mathcal{S}_m$ of germs of solutions of the equation $\mathcal{D}s = 0$ at a point m of V is upper-bounded by the number*

$$\sum_{h=0}^{h_0} \binom{n-1+h}{h} . \frac{p(n-1)-(q-p)h}{n-1+h} \quad \Big(= p.c(n+1,h_0) - q.c(n+1,h_0-1)\Big).$$

If D is moreover, calibrated, we define a tautological connection on the vector bundle $\mathcal{E} := R_{h_0-1}$, such that

Theorem 5. *The space $\mathcal{S}$ of solutions of $(*)$ is isomorphic to the space of sections σ of $\mathcal{E}$ whose covariant derivative $\nabla\sigma$ vanishes. Hence, the dimension of $\mathcal{S}_m$ is maximal iff the curvature of this connection vanishes.*

We then apply these results by building, for any curvilinear d-web on V $(d > n)$, a linear differential operator which is *always ordinary and calibrated*, and for which *solutions of* $(*)$ *are the* $(n-1)$*-abelian relations* of the web. After theorem 1, we recover the upper-bound already given by Damiano ([3]), for the rank of the web (dimension of the space of $(n-1)$-abelian relations), by taking $p = d - n$ and $q = d - 1$ (hence $h_0 = d - n$).

As a corollary of Theorem 5, we can define for any curvilinear web a notion of "curvature", and prove :

Theorem 6. *The Damiano's upper-bound for the rank of a curvilinear web is reached iff its curvature vanishes.*

The main interest of this result is that it is no more necessary to exhibit the abelian relations for proving the maximality of the rank.

Such a definition for the curvature of a web, whose vanishing is equivalent to the maximality of the rank, goes back to Blaschke ([1]) in the case $n = 2, d = 3$. Various generalizations have been done since, mainly for planar webs ([6, 8, 9]), for webs of codimension one ([2, 4]) when they are "ordinary", and for d-curvilinear webs when $d = n + 1$ ([5]). The definition that we give below for d-curvilinear webs whatever be d is new ; it has been announced in a preprint ([7]).

As an example, we prove (using a computation with Maple) that *the curvature of the exceptional* 6*-web* $W_{0,6}$ *in dimension 3 vanishes*, hence recovering that it has has a maximal rank (as well as his 4 and 5-subwebs). In fact all $n+3$-webs $W_{0,n+3}$ (that we re-defined here in words of vector fields) have a maximal rank : this has been claimed by Damiano ([3]) for any n, and proved by him for n even. He made a mistake in the proof for n odd, which has been corrected by Pirio ([10]).

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Persistent interaction topology in data analysis

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Abstract. Topological data analysis, as a tool for extracting topological features and characterizing geometric shapes, has had tremendous success across diverse fields. Its key mathematical techniques include persistent homology and the recently developed persistent Laplacians. However, classic mathematical models like simplicial complexes often struggle to provide a localized topological description for interactions or individual elements within a complex system involving a specific set of elements. In this work, we introduce persistent interaction homology and persistent interaction Laplacian that emphasize individual interacting elements in the system. We demonstrate the stability of persistent interaction homology as a persistent module. Furthermore, for a finite discrete set of points in the Euclidean space, we provide the construction of persistent interaction Vietoris-Rips complexes and compute their interaction homology and interaction Laplacians. The proposed methods hold significant promise for analyzing heterogeneously interactive data and emphasizing specific elements in data. Their utility for data science is demonstrated with applications to molecules.

Reeb vector field as isometric embedding

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Considering **Sasaki metric** on the unit tangent bundle T_1M, a map $\xi : (M, g) \to (T_1M, G)$, defining by $\xi(x) = (x, \xi(x))$, is **isometric embedding** only if ξ is **parallel**. The rigidity of Sasaki metric motivates many authors consider various deformations of Sasaki metric (see [1, 5, 6, 7, 11]). In particular, Domenico Perrone [8] studied **Reeb vector fields** with respect to a **Riemannian g-natural metrics** on the unit tangent bundle.

A **Riemannian** g**-natural metric** [1, 2] on the unit tangent bundle T_1M is defined by

$$\begin{aligned} G_{(x,\xi)}(X^h, Y^h) &= (a+c)g_x(X,Y) + dg_x(X,\xi)g_x(Y,\xi), \\ G_{(x,\xi)}(X^h, Y^v) &= bg_x(X,Y), \\ G_{(x,\xi)}(X^v, Y^v) &= ag_x(X,Y), \end{aligned}$$

where $a, b, c, d = const$, $a > 0$.

A **contact metric manifold** is defined by a set (M, g, η, ξ, ϕ), where M is a differential $2n+1$-dimensional manifold, ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is 1-form satisfying

$$\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$
$$(d\eta)(X, Y) = g(X, \phi Y).$$

Moreover, ξ is called **Reeb vector field** and η uniquely defines ξ by the conditions

$$\eta(\xi) = 1, \quad d\eta(\xi, X) = 0.$$

The Reeb vector field of a contact manifold plays a fundamental role in the study of the Riemannian geometry of a contact metric manifold (see [3]). A contact metric manifold (M, g, η, ξ, ϕ) is called **K-contact manifold** if ξ is **Killing vector field**. Moreover, if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$$

then the contact metric manifold (M, g, η, ξ, ϕ) is called **Sasakian manifold**.

Domenico Perrone [8] showed that there are **non-parallel** unit vector fields which define **isometric embeddings** with respect to a family of **Riemannian g-natural metrics** on the unit tangent bundle that depend on two parameters, which does not include the Sasaki metric.

Proposition 1. *Let (M, g, η, ξ, ϕ) be a contact metric manifold, $dimM = 2n + 1$, and let G be a Riemannian g-natural metric on T_1M with $c = 1 - 2a$. Then the map $\xi : (M, g) \to (T_1M, G)$ is an isometric embedding if and only if $d = a$ and M is K-contact manifold.*

If $\xi(x)$ is a unit vector field on M, then it defines a map $\xi : M \to T_1M$, defining by $\xi(x) = (x, \xi(x))$. From geometrical viewpoint $\xi(M)$ is explicitly given submanifold in T_1M.

A unit vector field ξ is said to be **harmonic** (see [9]) if it is a critical point of the energy functional defined on the space of all unit vector fields. The corresponding map $\xi : M \to T_1M$ is said to be **harmonic map** if it is a critical point of the energy functional defined on the space of all maps from M to T_1M. Note that a harmonic vector field ξ does not define, in general, a harmonic map from $\xi : M \to T_1M$.

Minimal submanifold is a submanifold with vector of mean curvature zero. A unit vector field ξ on Riemannian manifold M is called **minimal** (see [4]) if the image of (local) embedding $\xi : M \to T_1M$ is minimal submanifold in the unit tangent bundle T_1M. Note that an isometric immersion is minimal if and only if it is a harmonic map.

Domenico Perrone [8] suggested the following theorems.

Theorem 2. *The Reeb vector field ξ of a K-contact manifold (M, g, η, ξ, ϕ) defines a harmonic map $\xi : (M, g) \to (T_1M, G)$ for any Riemannian g-natural metric G on T_1M.*

Theorem 3. *Let (M, g, η, ξ, ϕ) be a K-contact manifold. Let $\mathcal{F}$ be the family of all Riemannian g-natural metrics on T_1M defined by the parameters*

$$0 < a < 1, \quad b^2 < a(1 - a), \quad c = 1 - 2a, \quad d = a.$$

Then, the Reeb vector field determines a minimal isometric immersion $\xi : (M, g) \to (T_1M, G)$ for any $G \in \mathcal{F}$.

Totally geodesic submanifold is a submanifold such that all geodesics in the submanifold are also geodesics of the surrounding manifold. A unit vector field ξ on Riemannian manifold M is called **totally geodesic** (see [10]) if the image of (local) embedding $\xi : M \to T_1M$ is totally geodesic submanifold in the unit tangent bundle T_1M. The corresponding map $\xi : M \to T_1M$ is said to be **totally geodesic map**. Namely, ξ is total geodesic if the second fundamental form of the map $\xi : M \to T_1M$ vanishes. Note that every totally geodesic map $\xi : M \to T_1M$ is harmonic and minimal.

The concept of totally geodesicity arises naturally in connection with the concepts of harmonicity and minimality. As a result, we have the following theorem.

Theorem 4. *Let (M, g, η, ξ, ϕ) be a K-contact metric manifold, $dim M = 2n+1$, and let G be a Riemannian g-natural metric on T_1M with $c = 1-2a$ and $d = a$. Then the Reeb vector field ξ defining the isometric embedding $\xi : (M, g) \to (T_1M, G)$ is totally geodesic if and only if M is Sasakian manifold.*

Thus totally geodesic property of the Reeb vector fields as isometric embeddings is distinguished Sasakian manifold among K-contact metric manifold with the Riemannian g-natural metrics on the unit tangent bundle.

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Classification of smooth structures on line with two origins

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We classified differentiable structures on a line $\mathbf{L}$ with two origins begin a non-Hausdorff but T_1 one-dimensional manifold obtained by "doubling" 0.

Definition 1. Let τ be the standard topology on $\mathbb{R}$. Then $\mathbf{L} = \mathbb{R} \sqcup \bar{0}$ is a disjoint union of $\mathbb{R}$ with some point $\bar{0}$ endowed with the follwoing topology:

$$\eta = \tau \cup \{(W \setminus 0) \cup \{\bar{0}\} : 0 \in W \in \tau\}$$

whose elements are elements of τ and also open neighborhoods of 0 in which 0 is replaced with $\bar{0}$.

For $k \in \mathbb{N} \cup \{\infty\}$ let H^k be the group of homeomorphisms h of $\mathbb{R}$ such that $h(0) = 0$ and the restriction of h $\mathbb{R} \setminus 0$ is a $\mathcal{C}^k$-diffeomorphism. It contains a subgroup D^k consisting of $\mathcal{C}^k$-diffeomorphisms of $\mathbb{R}$ also fixing 0.

Definition 2. Let H be a group and C, D be two subgroups. Then for each $h \in H$ the following subset of H:

$$ChD = \{chd^{-1} : c \in C, d \in D\}$$

is called the (C, D)-coset of h. If $C = D$, then DhD is also called the D-*double coset* of h. The set

$$Dh^{\pm 1}D := DhD \cup Dh^{-1}D = \{chd^{-1}, ch^{-1}d^{-1} : c, d \in D\}$$

is called the $(D, \pm)$-*double coset of* h.

We are refering to the book by J. Lee [1] and paper of F. Takens [2] for the definition of $\mathcal{C}^k$-structures. So the problem of classification of smooth $\mathcal{C}^k$-structures on $\mathbf{L}$ can be stated as follows:

Problem 3. Describe the orbits of the action of the group $\mathcal{H}(\mathbf{L})$ on the set of $\mathcal{C}^k$-structures on M.

It is shown that there is a natural bijection between $\mathcal{C}^k$-structures on $\mathbf{L}$ up to a $\mathcal{C}^k$-diffeomorphism and double coset classes $D^k \setminus H^k / D^k$ which can be regarded as the orbit space of the action $D^k \times D^k$ on H^k by the rule $(a, b)h = ahb^{-1}$.

Theorem 4. *Let* $k \in \mathbb{N} \cup \{\infty\}$. *Then*

- $\mathcal{C}^k$*-structures on* $\mathbf{L}$ *up to a* $\mathcal{C}^k$*=diffeomorphism are in one-to-one correspondence with the set* $\mathcal{D}(\mathbb{R}, 0) \setminus \mathcal{H}^k_0(\mathbb{R})^{\pm 1}/\mathcal{D}(\mathbb{R}, 0)$ *of* $(\mathcal{D}(\mathbb{R}, 0), \pm)$*-double coset classes;*
- *while* $\mathcal{C}^k$*-structures on* $\mathbf{L}$ *up to a* $\mathcal{C}^k$*-diffeomorphism fixing* 0 *and* $\bar{0}$ *are in one-to-one correspondence with the set* $\mathcal{D}(\mathbb{R}, 0) \setminus \mathcal{H}^k_0(\mathbb{R})/\mathcal{D}(\mathbb{R}, 0)$ *of* $\mathcal{D}(\mathbb{R}, 0)$*-double coset classes.*

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Vector bundle construction via monads on multiprojective spaces

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In this paper we construct indecomposable vector bundles associated to monads on multiprojective spaces. Specifically we establish the existence of monads on $\mathbf{P}^{2n+1} \times \mathbf{P}^{2n+1} \times \cdots \times \mathbf{P}^{2n+1}$ and on $\mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$. We prove stability of the kernel bundle which is a dual of a generalized Schwarzenberger bundle associated to the monads on $X = \mathbf{P}^{2n+1} \times \mathbf{P}^{2n+1} \times \cdots \times \mathbf{P}^{2n+1}$ and prove that the cohomology vector bundle which is simple, a generalization of special instanton bundles. We also prove stability of the kernel bundle and that the cohomology vector bundle associated to the monad on $\mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$ is simple. Lastly, we construct the morphisms that establish the existence of monads on $\mathbf{P}^1 \times \cdots \times \mathbf{P}^1$.

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About one problem of the Gauss-Kuzmin type

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Let $[a_0; a_1, ..., a_n, ...]$ — continued fraction, where $a_0 \in Z_+$, $a_j > 0$ for all $j \in N$. Consider left shift operator

$$T([0; a_1, a_2, ..., a_n, ...]) = [0; a_2, a_3, ..., a_{n+1}, ...].$$

Let $f_n(x) = \lambda(T^{-n}((0; x]))$, where $x \in (0; 1]$, $\lambda(\cdot)$ — Lebesgue measure. The problem of finding

$$f(x) = \lim_{n \to +\infty} f_n(x)$$

for classical continued fractions was posed by Gauss. Kuzmin [1] showed that $f(x) = \log_2(x+1)$ and

$$|f_n(x) - \log_2(1+x)| \leq C\beta^{(n^\eta)} \quad \forall x \in (0; 1]$$

for $\eta = 0,5$ some $C > 0$ and $\beta \in (0; 1)$. Levy [2] showed that it is possible to take $\beta = 0,7$ and $\eta = 1$. Wirsing [4] showed that, for the constant $\gamma \approx 0,3037$

$$\psi(x) = \lim_{n \to +\infty} \frac{f_n(x) - \log_2(1+x)}{(-\gamma)^n} \quad \forall x \in (0; 1],$$

where $\psi(x)$ — analytic function.

It is known [3] that for each $t \in [0,5; 1]$ there exists a sequence (b_n) such that $b_n \in \{0,5; 1\}$ for all $n \in N$ and $t = [0; b_1, ..., b_n, ...]$. The last image is called A_2-image. A countable set of numbers $t \in [0,5; 1]$ has two A_2-images.

Theorem 1. *For A_2-image the following conditions are true for some numbers $C_1 > C_2 > 0$ and for each $n \in N$*

$$|f_{n+1}(x_1)-f_{n+1}(x_2)| = |f_n((1+x_1)^{-1})-f_n((1+x_2)^{-1})|+|f_n((0,5+x_1)^{-1})-f_n((0,5+x_2)^{-1})| \forall x_1, x_2 \in [0,5;1];$$

$$C_2|x_2 - x_1| \leq |f_n(x_1) - f_n(x_2)| \leq C_1|x_2 - x_1| \ \ \forall x_1, x_2 \in [0,5;1], x_1 \leq x_2.$$

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Homotopy types of stabilizers of Morse-Bott functions on 3-manifolds

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Let M be a smooth 3-manifold, $\mathcal{D}(M)$ be the group of all C^∞ diffeomorphisms of M. For every smooth function $f : M \to \mathbb{R}$ denote by

$$\mathcal{S}(f) = \{h \in \mathcal{D}(M) \mid f \circ h = f\}$$

the stabilizer of f with respect to the natural action of $\mathcal{D}(M)$ on the space of all C^∞ functions on M. It consists of diffeomorphisms leaving invariant each level set of f. Endow $\mathcal{S}(f)$ with the corresponding strong C^∞ Whitney topology.

Let B be a submanifold of M. Then a *regular neighborhood* of B is a vector bundle $p\colon E \to B$ defined on an open neighborhood E of B in M and being a smooth retraction onto B. In that case a function $g\colon E \to \mathbb{R}$ is called 2-*homogeneous* if $g(tx) = t^2 g(x)$ for all $x \in E$ and $t \geq 0$.

Definition 1. Say that a Morse-Bott function $f : M \to \mathbb{R}$ is 2-*homogeneous* if for every critical submanifold B of f of dimension 1 and 2 there exists a tubular neighborhood $p\colon E \to B$ and a 2-homogeneous on fibers function $g\colon E \to \mathbb{R}$ such that $f = g$ near B.

Notice that in general (due to Morse-Bott lemma) f is 2-homogeneous only locally at each critical point x of f.

Now let $f\colon M \to \mathbb{R}$ be a C^∞ Morse-Bott function taking constant values at boundary components of M. Let also Γ be the Kronrod-Reeb graph of f, being the quotient of M by the partition into connected component of every level set of f, and $p : M \to \Gamma$ be the natural projection.

Say that an edge e of Γ is *internal* if its vertices have degrees ≥ 2, i.e. they correspond to non-extremal critical submanifodls of f. At each edge e of Γ fix a point x_e and put $N_e = p^{-1}(x_e)$. Thus, N_e is a closed subsurface of M on which f takes a constant value.

Theorem 2. *Let $f\colon M \to \mathbb{R}$ be a 2-homogeneous Morse-Bott function. Let also n be the total number of those N_e for which*

- *the edge e is internal and*
- *N_e is a 2-sphere or a projective plane.*

Then the higher homotopy groups of $\mathcal{S}(f)$ are n-powers of the corresponding 1-times higher homotopy groups of 2-sphere:

$$\pi_k\mathcal{S}(f) = \underbrace{\pi_{k+1}S^2 \times \cdots \times \pi_{k+1}S^2}_{n}, \quad k \geq 2.$$

In particular, if there are no such spheres and projective spaces, then $\mathcal{S}(f)$ is aspherical.

Elliptic Virtual Structure Constants and Generalizations of BCOV-Zinger Formula to Projective Fano Hypersurfaces

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In this talk, we propose a recipe for B-model computation of genus 1 Gromov-Witten invariants of Calabi-Yau and Fano Projective Hypersurfaces. Our formalism can be applied equally to both Calabi-Yau and Fano cases. In Calabi-Yau case, drastic cancellation of terms used in our formalism occurs and it results in another representation of BCOV-Zinger formula for projective Calabi-Yau hypersurfaces.

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Deformation properties of smooth functions on Klein bottle

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Let M be a connected compact C^∞-smooth 2-manifold. If $X \subset M$ is a closed subset of M, then $\mathcal{D}(M, X)$ denotes the group of diffeomorphisms of M, which are identity on X, endowed with the strong Whitney topology. If $X = \varnothing$, we omit X from notation. K denotes Klein bottle.

Consider space $C^\infty(M, \mathbb{R})$ endowed with the strong Whitney topology. Then the following right action of $\mathcal{D}(M, X)$ on $C^\infty(M, \mathbb{R})$ is defined: $C^\infty(M, \mathbb{R}) \times \mathcal{D}(M, X) \to C^\infty(M, \mathbb{R})$, $(f, h) \longmapsto f \circ h$. For each $f \in C^\infty(M, \mathbb{R})$, let $\mathcal{S}(f, X)$, $\mathcal{O}(f, X)$ be the stabilizer and the orbit of f with respect to that action. Let $\mathcal{D}_{\mathrm{id}}(M,X)$, $\mathcal{S}_{\mathrm{id}}(f,X)$ and $\mathcal{O}_f(f,X)$ be respective connected components of $\mathcal{D}(M,X)$, $\mathcal{S}(f,X)$, $\mathcal{O}(f,X)$ containing denoted subscripts. Also, we use notation $\mathcal{S}'(f,X) = \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(M,X)$.

Proposition 1. *Let $f \in C^\infty(M,\mathbb{R})$ be such that every its germ in every its critical point is C^∞-equivalent to some homogeneous polynomial without multiple factors, and f is constant on the boundary components of M. Then there are three mutually exclusive possibilities:*

(a) *its Kronrod–Reeb graph Γ_f is acyclic, and there exists component α of some critical level set $f^{-1}(a)$ and open disks $D_1,\dots,D_m$ such that $K \setminus \alpha = \bigsqcup_{i=1}^m D_i$,*
(b) *Γ_f is acyclic, and there exists component β of some regular level set $f^{-1}(b)$ and open Möbius bands M_1, M_2 such that $K \setminus \beta = M_1 \bigsqcup M_2$,*
(c) *Γ_f has a cycle, and there exists component C of some regular level set $f^{-1}(c)$, corresponding to a point on the cycle, and open cylinders $Q_1,\dots,Q_m$ such that $K \setminus \{h(C) \mid h \in \mathcal{S}(f)\} = \bigsqcup_{i=1}^m Q_i$.*

Theorem 2. *In the case (b) of Proposition 1 there is an isomorphism*

$$\pi_1\mathcal{O}_f(f) \cong \pi_0\mathcal{S}(f|_{M_1}, \partial M_1) \times \pi_0\mathcal{S}(f|_{M_2}, \partial M_2).$$

For Mobius band M group $\pi_0\mathcal{S}(f|_M, \partial M)$ was computed in [1].

Let $C \subset K$ be a closed curve, that corresponds to point on the cycle of Γ_f. Let Q be the cylinder bounded by C and the next curve among $\{C_1 \equiv C, C_2, \dots, C_m\} = \{h(C) \mid h \in \mathcal{S}(f)\}$. Denote $G = \pi_1\mathcal{O}(f|_Q, \partial Q)$, and let $G \wr_{m,\gamma} \mathbb{Z}$ be certain type of wreath product depending on γ.

Theorem 3. *In the case (c) of Proposition 1 there are two possibilities:*

(i) *either for every $h \in \mathcal{S}(f)$ equality $h(C) = C$ implies that h preserves orientation of C.*
Then there is an isomorphism

$$\pi_1\mathcal{O}_f(f) \cong G \wr_m \mathbb{Z},$$

and m can be only odd,
(ii) *or there exists $h \in \mathcal{S}(f)$ such that $h(C) = C$ and h changes orientation of C.*
Then exists an automorphism $\gamma\colon G \to G$ with $\gamma^2 = \mathrm{id}$ such that there is an isomorphism

$$\pi_1\mathcal{O}_f(f) \cong G \wr_{m,\gamma} \mathbb{Z}.$$

Theorem 4. *Consider composition $T^2 \xrightarrow{\pi} K \xrightarrow{f} \mathbb{R}$, where $f \in C^\infty(M,\mathbb{R})$ is the same as in Proposition 1, and π is the orientable double covering of Klein bottle with the torus. Then there are subgroups $\pi_0\mathcal{S}'(f) \hookrightarrow \pi_0\mathcal{S}'(f \circ \pi)$ and $\pi_1\mathcal{O}_f(f) \hookrightarrow \pi_1\mathcal{O}_{f\circ\pi}(f \circ \pi)$. Particularly, considering the respective cases of Theorem 3 holds the following:*

(i) $\pi_1\mathcal{O}_f(f) \cong G \wr_m \mathbb{Z} \hookrightarrow G^2 \wr_m \mathbb{Z} \cong \pi_1\mathcal{O}_{f\circ\pi}(f \circ \pi)$,
(ii) $\pi_1\mathcal{O}_f(f) \cong G \wr_{m,\gamma} \mathbb{Z} \hookrightarrow G \wr_{2m} \mathbb{Z} \cong \pi_1\mathcal{O}_{f\circ\pi}(f \circ \pi)$.

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Algebraic periods of surface homeomorphisms

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A self-map $f\colon M \to M$ of a compact manifold determines the sequence $\{L(f^n)\}$, $n \geq 1$, of the Lefschetz numbers of its iterations. We consider its dual sequence $\{a_n(f)\}_{n=1}^{\infty}$ given by the Möbius inversion formula. The set $\mathcal{AP}(f) = \{n : a_n(f) \neq 0\}$ is called the set of algebraic periods of f. During the talk we describe finite sets of algebraic periods of homeomorphisms of an orientable surface, especially of Morse–Smale diffeomorphisms.

The talk is based on the joint project with G. Graff, W. Marzantowicz and A. Myszkowski.

Wigner-Ville distribution associated with quadratic Clifford-Fourier transform

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This presentation provides the general double-sided orthogonal 2n-1-dimensional spaces split quadratic phase Clifford-Fourier transform and the general Wigner-Ville Distribution quadratic phase Clifford-Fourier transform. It proves the Rènyi and Shannon entropy and Lieb's uncertainty principles.

Keywords: Clifford Algebra; Orthogonal 2D-Planes Split; Clifford-Fourier Transform; Quadratic Phase Clifford-Fourier Transform; Wigner-Ville dDistribution Quadratic Phase Clifford-Fourier Transform.

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Non-simple strongly nilpotent distribution germs

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All known to-date Goursat structures featuring moduli of the local classification appear not to be strongly nilpotent. That is, not being equivalent to their nilpotent approximations. In the course of obtaining more and more (if disparate) confirmations of that statement, like in [1] and [2], a natural problem in the nonholonomic analysis has been gradually imposing by itself: to find a concrete unimodal family of strongly nilpotent completely nonholonomic distributions. The search for such an example seems by now hopeless in the realm of Goursat structures *per se*. Yet, by a neat perturbation of rank-two Goursat distributions in the underlying dimension 7 we obtain now a 1-parameter family of strongly nilpotent pairwise nonequivalent distribution germs. That family is given in local coordinates that happen to be already adapted. The members of the family are all quasi-homogeneous with respect to the weights defined by the small growth vector, one and the same for all members of the family. This property automatically yields their strong nilpotency, and also facilitates a proof of their pairwise non-equivalence.

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Extending of partial metrics

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A function $p : X^2 \to [0, +\infty)$ is called *a partial metric* on X if for every $x, y, z \in X$ the following conditions

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;
(p_2) $p(x, x) \leq p(x, y)$;
(p_3) $p(x, y) = p(y, x)$;
(p_4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

are true.

For any partial metric $p : X^2 \to [0, +\infty)$ the function $q_p : X^2 \to [0, +\infty)$, $q_p(x, y) = p(x, y) - p(x, x)$, is a quasi-metric on X and the topology of the partial metric space (X, p) is the topology τ_q of the quasi-metric space (X, q_p). Moreover, the function $d_p : X^2 \to [0, +\infty)$, $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ is a metric on X.

The following theorem was proved by F. Hausdorff in 1930.

Theorem 1. *Let X be a metrizable space, $A \subseteq X$ be a closed subset and $d_A : A^2 \to \mathbb{R}$ be a compatible metric on A. Then there exists a compatible metric $d : X^2 \to \mathbb{R}$ on X such that $d|_{A^2} = d_A$.*

Problem 2. Let X be a partial metrizable space, $A \subseteq X$ be a closed subset and $p_A : A^2 \to \mathbb{R}$ be a compatible partial metric on A. Does there exist a compatible partial metric $p : X^2 \to \mathbb{R}$ on X such that $p|_{A^2} = p_A$?

Proposition 3. *Let (X, p) be a partial metric space. Then the function $f : X \to \mathbb{R}$, $f(x) = p(x, x)$, is an 1-Lipschitz function with respect to the metric d_p.*

Proposition 4. *Let (X, d) be a metric space and $f : X \to [0, +\infty)$ be an 1-Lipschitz function. Then the function $p : X^2 \to \mathbb{R}$,*

$$p(x, y) = \tfrac{1}{2}(d(x, y) + f(x) + f(y)),$$

is a partial metric on X such that $d = d_p$ and $p(x, x) = f(x)$ for every $x \in X$.

Theorem 5. *Let X be a quasi-pseudometrizable space, A be a closed subset of X and $q_A : A^2 \to \mathbb{R}$ ba a compatible bounded quasi-pseudometric on A. Then there exists a compatible quasi-pseudometric q on X such that $q|_{A^2} = q_A$.*

Corollary 6. *Let X be a partial metrizable space, A be a closed subset of X and $p_A : A^2 \to \mathbb{R}$ ba a compatible partial metric on A such that q_{p_A} is bounded. Then there exists a compatible partial metric p on X such that $p|_{A^2} = p_A$.*

Proposition 7. *There exist a quasi-pseudometric space (X, q), a τ_q-closed set $A \subseteq X$ and a quasi-pseudometric p on A such that*
(1) q and p are equivalent on A;
(2) $\tau_r \not\subseteq \tau_q$ for every extension r of p on X.

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Axiomatic Development of Complexity Theory for Finite Groups

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> *"In any field of mathematics, the study of complexity is the first level of sophistication beyond knowing the building blocks."* – John Rhodes

What are the simplest ways to construct a finite group from its atomic constituents? To understand part-whole relations between finite simple groups ('atoms') and the global structure of finite groups, we axiomatize complexity measures on finite groups. From the Jordan-Hölder theorem and Frobenius-Kalužnin-Krasner-Lagrange embedding in an iterated wreath product, any finite group G can be constructed from a unique collection of simple groups, its Jordan-Hölder factors, each with well-defined multiplicities through iterated extension by simple groups. What is the least number of levels needed in such a hierarchical construction if a level is allowed to include several of these atomic pieces? To pose and answer this question rigorously, we give a natural set of hierarchical complexity axioms for finite groups relating to constructability, extension, quotients, and products, and prove these axioms are satisfied by a unique maximal complexity function **cx**. We prove this function is the same as the minimal number of "spans of gems" or "completely reducible groups" (i.e., direct products of simple groups) in a subnormal series with all factors of this type. Hierarchical complexity **cx** is thus effectively computable, and bounded below by all other complexity measures satisfying the axioms, including generalizations of derived length, Fitting height and solvability. Also, the hierarchical complexity of a normal subgroup is bounded above by the complexity of the whole group, although this is not assumed in the axioms and does not follow from the axioms for general (non-maximal) complexity functions satisfying the axioms.

For solvable finite groups, the unique maximal group complexity measure satisfying the axioms on this class agrees with the restriction of the one for all finite groups, and in addition satisfies an embedding axiom - which decidedly cannot be applied in the general case of all finite groups. In both cases, the complexity of a group is bounded above and below by various natural functions. In particular, hierarchical complexity is sharply bounded above by socle length, which yields a canonical decomposition and satisfies all the axioms except the extension axiom. Examples illustrate applications of the bounds and axiomatic methods in determining complexity of groups. We show also that minimal decompositions need not be unique in terms of what components occur nor their ordering. The complexity axioms are also shown to be independent.

Construction and application of quasicrystals

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We discuss three methods that generate n-dimensional quasicryatals and propose two applications of quasicrystals to data processing. The first application is to use the mapping between the physical and internal spaces of a quasi-crystal to evenly distribute data that is lost in the process of transmitting or storing information. At the same time, it is possible to unambiguously restore the rest of the data. The second application consists in the construction of special quasi-crystals that satisfy the requirements of keys of any length for the classical Vernam cipher method. Several examples of construction of quasicrystals with predetermined properties and examples of image processing that makes the loss of its part uniformly distributed are given.

Different physical phenomena arising from the interaction of incommensurate frequencies display the features of almost periodicity. A typical example is a potential field of a physical quasicrystal. Quasicrystals are discrete structures that have highly structured long-range order (represented by pure point or near pure point diffraction) but don't have periodic order. A standard approach to modeling such structures is to take a finite part of it, impose periodic boundary conditions and then apply usual crystallography. Although this type of periodization is used routinely and successfully for many modelling problems in the theory of quasicrystals, it is not entirely satisfactory. Almost periodic order goes beyond periodic order in fundamental ways, its essence appearing as a underlying incommensurability which pervades every part of the theory.

It is possible to construct quasicrystals by means of tiling, fractals and cut-and-project method. The general idea of cut-and-project method is shown in the scheme

$$\begin{tikzcd}
\mathbb{R}^d & \mathbb{R}^d \times \mathbb{R}^d \arrow[l, "||"'] \arrow[r, "\perp"] & \mathbb{R}^d \\
 & \cup & \\
L & \widetilde{L} \arrow[l, "1-1"'] \arrow[r, "\text{dense image}"] & L'
\end{tikzcd}$$

Here $\widetilde{L}$ is a lattice in $\mathbb{R}^d \times \mathbb{R}^d$ which is oriented so that the projections into $\mathbb{R}^d$ are $1-1$ and dense.

The left-hand $\mathbb{R}^d$ is *physical space* (where quasicrystal Λ lies).

The right-hand $\mathbb{R}^d$ is *internal space* (to control the projection).

$\tilde{x} \in \widetilde{L}$, $\tilde{x} = (x, x')$ where $x \in L$ and $x' \in L'$.

$(\,\cdot\,)' \;:\; L \to L'$ is defined by $x \mapsto x'$, which passes from physical to internal space.

Window Ω is chosen in internal space (compact, equal to the closure of its interior, and have boundary of measure 0).

Quasicrystal Λ can be defined in the following way $\Lambda(\Omega) := \{x \mid \tilde{x} \in \widetilde{L},\, x' \in \Omega\}$.

We applied the transformation $(\cdot)'$ to bitmaps and using its discontinuity property we can distribute the lost information evenly throughout the image.

Considering one-dimensional quasi-crystals, we established that some of them can serve as binary keys for the Vernam cipher, while the keys can have an arbitrary length and be uniquely constructed from a small number of integers, namely from the seed point, window length and integer coefficients of a quadratic equation.

Figure 0.1. An example of information loss during data transmission in the original raster image and in the image encoded with the help of a quasi-crystal.

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On Beloch's curve that appears when solving real cubic with origami

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Origami construction is generally defined by 7 Axioms (re-)founded by Justin–Huzita–Hatori in 1980's (cf. [Jus86, p.40–45], [Lan10],[Ned22]). Among other things, Axiom 6 allows so-called "neusis", which is not allowed in the straightedge and compass construction.

1 (Justin–Huzita–Hatori Axioms 6). *Given two points A and B and two lines l_1 and l_2 on a plane, there exists a fold that places A onto l_1 and B onto l_2 at the same time. In other words, it is possible to construct a certain fold line l such that points symmetrical to points A and B with respect to l are placed on the straight lines l_1 and l_2, respectively.*

This Axiom 6 enables to construct all solutions of any given real cubic equation by using a perfect piece of origami. The following method was initially shown by Beloch in 1936 [Bel36], based on Lill's enjoyable idea [Lil67] (see also [Kat99], [Hul11], [NO15]).

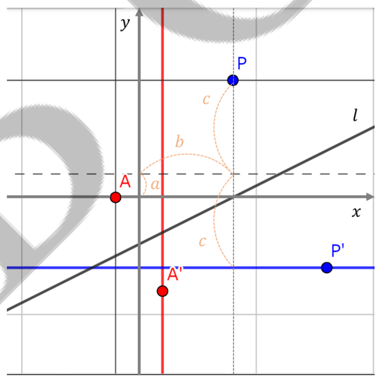
Theorem 2. *Given segments with length $a, b, c \in \mathbb{R}$, every solution of the cubic formula*

$$x^3 - ax^2 - bx + c = 0 \quad \cdots \quad (*)$$

may be constructed. More precisely,

(1) If we make a fold that places (I) $A(-1, 0)$ onto $x = 1$ and (II) $P(b, a + c)$ onto $y = a - c$, then the y-intercept r of the fold line l is a solution of $(*)$

(2) For any solution $x = r$ of $(*)$, a fold that places $A(-1, 0)$ onto $A'(1, 2r)$ satisfies the conditions (I) and (II) in (1).



In other words, *there is a bijective correspondence between all fold lines satisfying the conditions* (I) and (II) *and all real solutions of* $(*)$. In addition, by considering all fold lines satisfying (I) and parametrizing them by r, we may find all solutions.

We consider all fold lines satisfying the condition (I), and investigate the orbit of the points $P'(r)$ ($r \in \mathbb{R}$), each of which is symmetric to P with respect to a fold line. We set $(p, q) = (b, a + c)$. Our main results may be summarized into the following theorem.

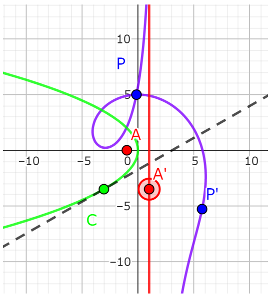
Theorem 3. *The union $\mathcal{F}$ of the orbit of the points P' and the point $P(p, q)$ is a real cubic curve*

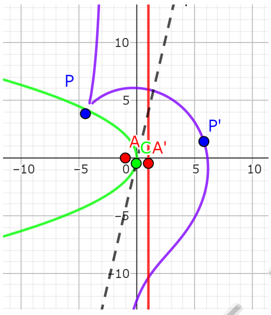
$$F(x, y) := 2(q - y)^2 - (q + y)(q - y)(p - x) - (p - x)^2(p + x) = 0,$$

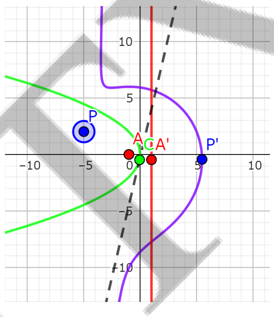
which we call Beloch's curve.

The point P *is its uniquely existing singular point, and the Hessian at* P *is given by* $\mathcal{H}_{\mathcal{F}} = -4(4p+q^2)$. *We have the following equivalence on the shape of* $\mathcal{F}$ *and the parabola* $\mathcal{G} : 4x+y^2 = 0$.

- P *is on the left side of* $\mathcal{G}$ $\iff$ *The orbit of* P'*'s does not pass through* P $\iff$ P *is an isolated point of* $\mathcal{F}$.
- P *is on* $\mathcal{G}$ $\iff$ *The orbit of* P'*'s passes through* P *just once* $\iff$ P *is a cusp of* $\mathcal{F}$.
- P *is on the right side of* $\mathcal{G}$ $\iff$ *The orbit of* P'*'s passes through* P *twice* $\iff$ P *is a self-intersection point of* $\mathcal{F}$.







We may also classify the shapes of real cubic curves $a_0y^2 - a_1xy^2 - a_3x^2 - a_4x^3 = 0$ in a similar manner. In addition, we show that the rotation number of $\mathcal{F}$ around the point A is determined by the relationship between P and $x = 1$.

In the proof of 3, we use the fact that the fold line l is the tangent line of the parabola $\mathcal{G}$ at $(-r^2, 2r)$. In general, given a point A, Q, and a line m, then Axiom 5 allows a fold l such that $Q \in l$ and l maps A onto m. 3 would shed new light on the relationship between Axioms 5 and 6.

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Hypercyclicity of symmetric composition operator

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The classical Birkhoff theorem (1929) [1] asserts that any operator of composition with translation

$$x \mapsto x + a,$$
$$T_a \colon f(x) \mapsto f(x + a)$$

is hypercyclic on the space of entire functions $H(\mathbb{C})$ on the complex plane $\mathbb{C}$ if $a \neq 0$. A generalization of the Birkhoff theorem was proved by Godefroy and Shapiro in [2].

Definition 1. Let X be a topological space. A continuous linear operator $T : X \to X$ is said to be *hypercyclic* if there is some vector $x \in X$ such that the set

$$\mathrm{Orb}(T, x) = \{x, Tx, T^2x, \ldots\}$$

of iterates of x is dense in X. The vector x is called a hypercyclic vector associated to the hypercyclic operator T.

The hypercyclicity of a special operator on an algebra of symmetric analytic functions on ℓ_1 was proved in [3]. We construct new class of hypercyclic composition operators on an algebra of symmetric analytic functions on ℓ_1.

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On (i, j)-Baire Bilocales

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ABSTRACT: In the category of bitopological spaces, a bitopological space (X, τ_1, τ_2) is said to be *almost* (i, j)*-Baire* [1] if every sequence $\{G_n : n \in \mathbb{N}\}$ of τ_j-open τ_i-dense subsets of X satisfies the condition that $\bigcap_{n\in\mathbb{N}} G_n$ is τ_i-dense, where $i, j = 1, 2$, $i \neq j$. In this talk, we transfer this notion of almost (i, j)-Baireness to bilocales. In our notion though, the prefix "almost" is dropped. So, we define and characterize (i, j)-Baire bilocales. We also give internal properties of (i, j)-Baire bilocales which are not translated from properties of almost (i, j)-Baireness in bitopological spaces. For instance, we show that in the class of Noetherian bilocales, (i, j)-Baireness of a bilocale

coincides with (i, j)-Baireness of its ideal bilocale. We also consider relative versions of (i, j)-Baire where we show that a bilocale is (i, j)-Baire only if the subbilocale induced by the Booleanization is (i, j)-Baire.

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Application of the dynamical system theory for counting black hole entropy of microstates

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Superstring theory is one of the most advanced theories in physics that attempts to unify all four fundamental forces of nature into one single theory. It is based on the idea that all elementary particles and forces in nature can be explained as vibrations of ultramicroscopic strings. Mathematical models in superstring theory have their own unique properties and applications. They make it possible to describe various physical phenomena and processes, such as gravity, electromagnetism, strong and weak nuclear interactions. One of the main properties of mathematical models in superstring theory is their geometric nature. They describe spacetime as a multidimensional space in which strings can move. This allows us to explain many properties of space-time, such as its curvature and topology. The use of mathematical models in superstring theory also makes it possible to study various physical phenomena and processes. For example, they can be used to describe the processes of birth and decay of elementary particles, as well as to explain the properties of black holes and other exotic objects. For example, the Schwarzschild model is used to describe the gravitational field of a black hole

$$ds^2 = -\left(1 - \frac{2MG}{r}\right)dt^2 + \frac{1}{\left(1 - \frac{2MG}{r}\right)}dr^2 + r^2 d\Omega_2^2.$$

This model allows us to describe the properties of a black hole, such as its radius, r, mass, M. Quantum gravity models are also used to explain the properties of black holes. For example, the loop quantum gravity model allows us to describe the properties of black holes at the microscopic level.

Combining quantum mechanics and thermodynamics leads to many hidden degrees of freedom that give a black hole its entropy. These degrees of freedom do not appear in the classical description of black holes and are associated with string theory. The entropy of a black hole from string theory was calculated by Susskind [1]. The calculations of the string entropy is realized through the consideration of a multidimensional lattice of points with the strings inside it, which can move in any of 2d directions. So, the string entropy is

$$S = ln(2d)^n = n \cdot ln2d.$$

Let us consider the use of mathematical models in the aspect of the theory of dynamical systems through the concept of topological entropy to describe chaotic behavior in dynamics, [2]. One can calculate the volume entropy of such space, B,

$$h_v \sim log(VolB)) \sim log(2d)^n.$$

The volume entropy h_v is always bounded above by the topological entropy h_{top} of the geodesic flow on M. Moreover, if M has non-positive sectional curvature, then

$$h_v = h_{top}.$$

From the other hand we know, that

$$h_{top} \geq log|deg(f)|.$$

As the fixed point theorem was proved in 1912 by Brouwer [3], so any continuous mapping of a sphere onto itself has isolated point. Mapping of spaces ($R^{2d} \to R^{2d}$) for ($d = 1, \ldots, n$) can be presented by $f : S^{2d} \to S^{2d}$ and determines the degree of mapping $degf = 2d$. Therefore, topological entropy of the system of n links of string length, $L = l_s n$, is the n sum,

$$h_{top} = n \cdot log(2d).$$

The Lefschetz number of one link

$$L(f^n) = 1 + (-1)^m degf^n$$

is equal to

$$L(f^n) = 1 + (2d)^n.$$

So, according to the Lefschetz formula we can calculate the index of isolated point on manifold M

$$L(f) = \sum ind_f x.$$

Considering the Hopf-Poincare theorem

$$\sum ind_f x = \chi(M)$$

we can receive the following formula

$$1 + ((2d)^n)^n = \chi(M).$$

Thus, using the theory of dynamical systems, we calculated the entropy, Lefschetz number and Euler characteristic of a black hole, represented as a multidimensional cubic space.

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Computing the Gromov–Hausdorf distance using gradient methods

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The Gromov–Hausdorff distance measures the difference in shape between metric spaces, which has been used in image processing and data analysis. Computing it can be formulated as a generalization of the NP-hard quadratic assignment problem without the bijectivity constraint. Continuous relaxations that are more tractable, such as the Gromov–Wasserstein distance, exist but fail to recover non-bijective solutions to the original problem. As a consequence, the approximation factor of the current relaxation algorithms is infinite in the general case.

We introduce a quadratic relaxation whose solutions provably deliver the Gromov–Hausdorff distance. The optimality guarantee is enabled in part by allowing non-bijections in the search space. We suggest a gradient method for approximating solutions to this relaxation, and show that it can efficiently compute the Gromov–Hausdorff distance between metric spaces of hundreds of points.

Magnetic trajectories on 2-step nilmanifolds

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From the mechanical perspective, the behaviour of a charged particle in presence of a force, known as Lorentz force, is described by an equation of the form:

$$\nabla_{\gamma'}\gamma' + F\gamma',$$

where γ is a differentiable curve on Riemannian manifold (M, g), ∇ is the corresponding Levi-Civita connection and F is a skew-symmetric $(1,1)$-tensor such that the corresponding 2-form $g(F\cdot,\cdot)$ is closed. Geodesics are the corresponding curves whenever $F \equiv 0$, that means that the particles do not experience any force. The magnetic trajectories are quite different from geodesics. For instance in the Euclidean plane, while geodesics are straight lines, magnetic trajectories are circles.

In this work, we concentrate the attention to magnetic trajectories associated to a left-invariant Lorentz force on a given 2-step nilpotent Lie group equipped with a left-invariant metric $(N, \langle\ ,\ \rangle)$ and the associated compact quotients. The main purposes are:

(i) to describe the solutions of the magnetic equation;
(ii) to determine closedness conditions of magnetic trajectories on compact quotients.

To facilitate the description of magnetic trajectories through the identity element, we make use of the structure of the Lie algebra. Any 2-step nilpotent Lie algebra with a metric admits an orthogonal decomposition

$$\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}, \qquad \mathfrak{v} = \mathfrak{z}^{\perp},$$

where $\mathfrak{z}$ denotes the center of $\mathfrak{n}$.

One proves that magnetic trajectories for left-invariant Lorentz forces preserving the decomposition above can be explicitly computed, see [1]. In other cases, the magnetic curves obey different features.

We shall show examples of magnetic trajectories. In particular on the Heisenberg Lie group of dimension three, one has examples related to elliptic integrals. As usual, once one finds the curves on the Lie group, one may induce them to compact quotients. In this situation, one search for closed magnetic trajectories.

On the other hand, we discuss obstructions to the existence of (left-invariant) Lorentz forces on 2-step nilpotent Lie algebras.

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N-foci balls in hyperbolic geometry

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Let's suppose that $\mathbb{H}^2 = \{(x, y) \mid y > 0\}$ is an upper half-plane with the Riemannian metric $\frac{dx^2 + dy^2}{y^2}$. It is called a hyperbolic plane and has a constant negative Gaussian curvature -1. Besides, $\mathbb{H}^2$ is a Hadamard space, which is a complete Riemannian manifold of nonpositive sectional curvature.

Between two any points $x, y \in \mathbb{H}^2$ there is a unique geodesic $\sigma_{x,y}$. So we can define a notion of a *geodesically convex* (or just *convex*) set in hyperbolic plane — it is a set that for two arbitrary points x and y of its $\sigma_{x,y}$ belongs to this set. Particularly, the mapping

$$\rho : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{R}, \rho(x, y) = \ell(\sigma_{x,y}), x, y \in \mathbb{H}^2,$$

where ℓ denotes a length of curve in $\mathbb{H}^2$, satisfies all the axioms of metric space.

We also can define a notion of convex function in $\mathbb{H}^2$.

Definition 1. We will call a parametrization $\gamma : [0, 1] \to \mathbb{H}^2$ of the geodesics between points a and b in $\mathbb{H}^2$, $\gamma(0) = a$, $\gamma(1) = b$, *standard*, if for all $\alpha \in (0; 1)$ the equality

$$\rho(a, \gamma(\alpha)) = \alpha\ell$$

holds. Here ℓ denotes a length of the appropriate geodesics.

Definition 2. A function $f : \mathbb{H}^2 \to \mathbb{R}$ is called *convex* in a convex set $A \subset \mathbb{H}^2$, if for arbitrary points $x_1, x_2 \in A$ and a standard parametrization $\gamma : [0, 1] \to \mathbb{H}^2$ of the geodesics between them, $\gamma(0) = x_2$, $\gamma(1) = x_1$, next inequality holds:

$$\forall\alpha \in [0, 1] : f(\gamma(\alpha)) \le \alpha f(x_1) + (1 - \alpha) f(x_2). \tag{1}$$

Definition 3. Let's fix in $\mathbb{H}^2$ any mutually distinct points $x_1, \dots, x_N$, where $N \in \mathbb{N}$, and such positive numbers $w_1, \dots, w_N, a$ that $\sum\limits_{k=1}^{N} w_k = 1$. *Open weighted N-foci ball*, or *weighted N-foci ball*, is a set

$$A = \{x \in \mathbb{H}^2 \,|\, w_1\, \rho(x, x_1) + \dots + w_N\, \rho(x, x_N) < a\}, \tag{2}$$

where $x_1, \dots, x_N$ are called *foci of the weighted N-foci ball*, a is called *a radius of the weighted N-foci ball*, $w_1, \dots, w_N$ are called *weights of the foci $x_1, \dots, x_N$*.

We can define closed weighted N-foci balls the same way, having replaced the symbol "$<$" by the symbol "$\le$" in the formula (2).

Let's fix any point $x_0 \in \mathbb{H}^2$ and define the distance function for it:

$$f : \mathbb{H}^2 \to \mathbb{R},\; f(x) = \rho(x, x_0),\; x \in \mathbb{H}^2.$$

Theorem 4. *The distance function f is convex in the hyperbolic plane $\mathbb{H}^2$.*

It is known, that such a function is convex in any Hadamard space [2]. In this work we got a direct proof of convexity of f for the case of the hyperbolic plane.

From the convexity of f we obtain another result.

Theorem 5. *All open and closed weighted N-foci balls are geodesically convex sets in the hyperbolic plane $\mathbb{H}^2$.*

We also proved geodesical convexity of 1-foci ball, which is a hyperbolic ball, with geometrical methods.

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A retraction from the space of pseudometrics to the space of ultrapseudometrics

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Definition 1. A **pseudometric** on a set X is a function $d : X \times X \to \mathbb{R}$ that satisfies the following properties for all $x, y, z \in X$:

(1) **Non-negativity**: $d(x, y) \geq 0$.
(2) **Identity of indiscernibles**: $d(x, x) = 0$. (However, it is not required that $d(x, y) = 0$ implies $x = y$, which differentiates a pseudometric from a metric.)
(3) **Symmetry**: $d(x, y) = d(y, x)$.
(4) **Triangle inequality**: $d(x, z) \leq d(x, y) + d(y, z)$.

An **ultrapseudometric** is a type of distance function defined on a set that generalizes the notion of a metric, incorporating properties specific to ultrametrics and pseudometrics. Formally:

Definition 2. An ultrapseudometric d on a set X is a function $d : X \times X \to \mathbb{R}$ that satisfies the above properties of **non-negativity, identity of indiscernibles, and symmetry**, but the **triangle inequality** is satisfied in a stronger form:

(4) **Strong triangle inequality (Ultrametric inequality)**[1] : for all $x, y, z \in X$

$$d(x,z) \leq \max\{d(x,y), d(y,z)\}$$

We denote with $\mathcal{P}\int(X)$ the set of all pseudometrics on a fixed set X, and $\mathcal{UP}\int(X)$ is its subset consisting of all ultrapseudometrics on X.

Theorem 3. *There is a non-expanding w.r.t. the uniform convergence metric retraction* $\mathcal{P}\int(X) \to \mathcal{UP}\int(X)$

We rely on the following lemmas.

Lemma 4. *For each pseudometric* $d : X \times X \to \mathbb{R}$ *and a subset* $A \subset X$ *the function* $d_A : X \times X \to \mathbb{R}$ *with the formula*

$$d_A(x,y) = \begin{cases} 0, & x, y \in A \text{ or } x, y \notin A, \\ d(A, X \setminus A), & x \in A, y \notin A \text{ or } x \notin A, y \in A, \end{cases} \qquad x, y \in X,$$

is an ultrapseudometric such that $d_A \leq d$.

Proof. Non-negativity, symmetry, and identity of indiscernibles clearly hold. The only not so trivial part is the strong triangle inequality.[2]

- If all three points are in A or all three are not in A: $d_A(x,z) = 0 \leq \max\{0,0\}$.
- If $x, y \in A$ and $z \notin A$ (or vice versa): $d_A(x,y) = 0$, $d_A(y,z) = d(A, X \setminus A)$, $d_A(x,z) = d(A, X \setminus A)$, hence $d_A(x,z) \leq \max\{0, d(A, X \setminus A)\}$.
- If $x \in A$, $y \notin A$, and $z \in A$ (or vice versa): $d_A(x,y) = d(A, X \setminus A)$, $d_A(y,z) = d(A, X \setminus A)$, $d_A(x,z) = 0$, hence $d_A(x,z) \leq \max\{d(A, X \setminus A), d(A, X \setminus A)\}$.

To compare d_A and d:

- If $x, y \in A$ or $x, y \notin A$: $d_A(x,y) = 0 \leq d(x,y)$.
- If $x \in A$ and $y \notin A$ (or vice versa): $d_A(x,y) = d(A, X \setminus A) \leq d(x,y)$.

□

Lemma 5. *For each pseudometric* $d : X \times X \to \mathbb{R}$ *the function* $\bar{d} : X \times X \to \mathbb{R}$ *such that*

$$\bar{d}(x,y) = \sup\{d_A(x,y) \mid A \subset X\}, \quad x, y \in X,$$

is the greatest ultrapseudometric on X *not exceeeding* d.

Proof. (1) **Ultrapseudometric properties of** $\overline{d}$**:**

(a) **Symmetry:**

$$\overline{d}(x,y) = \sup\{d_A(x,y) \mid A \subseteq X\} = \sup\{d_A(y,x) \mid A \subseteq X\} = \overline{d}(y,x)$$

since d_A is symmetric for all $A \subseteq X$.

(b) **Non-negativity and zero distance:**

$$\overline{d}(x,y) \geq 0$$

and

$$\overline{d}(x,x) = \sup\{d_A(x,x) \mid A \subseteq X\} = 0$$

since $d_A(x,x) = 0$ for all $A \subseteq X$.

(c) **Ultrametric inequality:** For all $x, y, z \in X$:

$$\overline{d}(x,z) = \sup\{d_A(x,z) \mid A \subseteq X\}$$

and

$$\overline{d}(x,z) \leq \sup\{\max\{d_A(x,y), d_A(y,z)\} \mid A \subseteq X\} \leq \max\{\overline{d}(x,y), \overline{d}(y,z)\}$$

since d_A satisfies the ultrametric inequality for all $A \subseteq X$.

(2) **Comparison $\overline{d} \leq d$:** For each $A \subseteq X$, we have $d_A \leq d$, thus:

$$\overline{d}(x,y) = \sup\{d_A(x,y) \mid A \subseteq X\} \leq d(x,y).$$

(3) **Greatest ultrapseudometric not exceeding d:** Suppose there exists another ultrapseudometric d' on X such that $d' \leq d$ and $d' \geq \overline{d}$. Then, for any $A \subseteq X$, $d_A \leq d'$, hence:

$$\overline{d} = \sup\{d_A \mid A \subseteq X\} \leq d'.$$

Therefore, $\overline{d}(x,y) = \sup\{d_A(x,y) \mid A \subseteq X\}$ is the greatest ultrapseudometric on X not exceeding d. □

We will discuss efficient algorithms for calculation of $\bar{d}$ for a given d on a finite set X.

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Action of derivations on polynomials and on Jacobian derivations

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Let $\mathbb{K}$ be an arbitrary field of characteristic zero. Denote by $A := \mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring, and by $R := \mathbb{K}(x_1, \ldots, x_n)$ the field of rational functions in n variables, respectively. A $\mathbb{K}$-linear map $D : A \to A$ is called a $\mathbb{K}$-derivation on A if $D(fg) = D(f)g + fD(g)$ for any $f, g \in A$. The vector space $W_n(\mathbb{K})$ (over $\mathbb{K}$) of all $\mathbb{K}$-derivation is a Lie algebra with respect to the Lie bracket $[D_1, D_2] = D_1D_2 - D_2D_1$, $D_1, D_2 \in W_n(\mathbb{K})$. Recall that every element $D \in W_n(\mathbb{K})$ can be uniquely written in the form

$$D = f_1\frac{\partial}{\partial x_1} + \cdots + f_n\frac{\partial}{\partial x_n}, f_i \in A.$$

The latter means that $W_n(\mathbb{K})$ is a free module of rank n over A with the free generators $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ (see, for example [3], [4]).

Every element D from $W_n(\mathbb{K})$ acts naturally on polynomials from A and on $W_n(\mathbb{K})$ itself (by multiplication). Recall that a polynomial $f \in A$ is a Darboux polynomial for a derivation $D \in W_n(\mathbb{K})$ if $D(f) = \lambda f$ for some $\lambda \in A$, the polynomial λ is called a cofactor for D. One can consider the Darboux polynomials as "eigenvectors" for the derivation D with polynomial "eigenvalues". These (non-constant) polynomials (if they do exist) play significant role in theory of differential equations because for a derivation $D = f_1\frac{\partial}{\partial x_1} + \cdots + f_n\frac{\partial}{\partial x_n}$ one can consider an autonomous system of differential equations of the form

$$\frac{dx_1}{dt} = f_1(x_1, \ldots, x_n), \ldots, \frac{dx_n}{dt} = f_n(x_1, \ldots, x_n)$$

and Darboux polynomials for D are very useful for searching solutions of this system (see, for example, [1], [2]).

We study normalizers of polynomials and derivations under the action of $W_n(\mathbb{K})$ on A and on itself (by multiplication) respectively. For any $f \in A$ one can consider the "normalizer" $N(f)$ in $W_n(\mathbb{K})$ of the form

$$N(f) = \{T \in W_n(\mathbb{K}) \mid T(f) = \lambda f \text{ for some } \lambda \in A\},$$

i.e. $N(f)$ is the set of all the derivations for which f is a Darboux polynomial. This normalizer is a subalgebra of the Lie algebra $W_n(\mathbb{K})$ and it acts on the principal ideal $(f) = Af$ of the ring A. The restriction $\widehat{N}(f)$ of the Lie algebra $N(f)$ on Af is characterized in the next statement.

Theorem 1. *The Lie algebra $\widehat{N}(f)$ is isomorphic to a subalgebra of the semidirect sum $W_n(\mathbb{K}) \leftthreetimes A$.*

Analogously for any $D \in W_n(\mathbb{K})$, one can consider the normalizer of D in $W_n(\mathbb{K})$ of the form

$$N(D) = \{T \in W_n(\mathbb{K}) \mid [T, D] = \lambda D \text{ for some } \lambda \in A\}$$

($N(D)$ is obviously the usual normalizer of the subalgebra AD in the Lie algebra $W_n(\mathbb{K})$). An analogous characterization of $N(D)$ is obtained.

Further, we consider more detailed the Lie algebra $W_2(\mathbb{K})$ and denote for convenience $A = \mathbb{K}[x, y]$. Let $f \in A$, $f \neq 0$. The polynomial f defines a derivation $D_f \in W_2(\mathbb{K})$ by the rule: $D_f(h) = \det J(f, h)$ for any $h \in \mathbb{K}[x, y]$ (here $J(f, h)$ is the Jacobi matrix for f and h). The derivation D_f is called the Jacobian derivation associated with the polynomial f. Note that all the Jacobian derivations form a subalgebra of $W_2(\mathbb{K})$ which coincides with the subalgebra $\mathfrak{sa}_2(\mathbb{K})$ consisting of all divergence-free derivations (see, for example [5]). If for some derivation $T \in W_2(\mathbb{K})$ there exists a Jordan chain consisting of polynomials

$$T(f_1) = \lambda f_1 + f_2, \ldots, T(f_{k-1}) = \lambda f_{k-1} + f_k, T(f_k) = \lambda f_k$$

for some $\lambda \in \mathbb{K}$, $k \geq 1$ then we prove the next statement

Theorem 2. *Let $T \in W_2(\mathbb{K})$ acts on polynomials $f_1, \ldots f_k$ by the rule*

$$T(f_1) = \lambda f_1 + f_2, \ldots, T(f_{k-1}) = \lambda f_{k-1} + f_k, T(f_k) = \lambda f_k$$

for some $\lambda \in \mathbb{K}$, $k \geq 1$. Then the equalities hold:

$$[T, D_{f_1}] = (\lambda - divT)D_{f_1} + D_{f_2},\ [T, D_{f_2}] = (\lambda - divT)D_{f_2} + D_{f_3}, \ldots,$$

$$[T, D_{f_k}] = (\lambda - divT)D_{f_k}.$$

The proof of this result is based on the next statement which is of independent interest.

Proposition 3. *Let $T \in W_2(\mathbb{K})$, $f \in \mathbb{K}[x, y]$ and $T(f) = g$ for some polynomial $g \in \mathbb{K}[x, y]$. Then $[T, D_f] = (-divT)D_f + D_g$. And conversely, if $[T, D_f] = (-divT)D_f + D_g$ for some $g \in A$, then $T(f) = g + c$ for some $c \in \mathbb{K}$.*

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Periodic point theorem for mappings contracting total pairwise distance

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We consider a new type of mappings in metric spaces so-called mappings contracting total pairwise distance on n points, see [1]. It is shown that such mappings are continuous. A theorem on the existence of periodic points for such mappings is proved and the classical Banach fixed-point theorem is obtained like a simple corollary as well as the fixed point theorem for mappings contracting perimeters of triangles.

Everywhere below by $|X|$ we denote the cardinality of the set X. Let (X, d) be a metric space, $|X| \geqslant 2$, and let x_1, x_2, ..., $x_n \in X$, $n \geqslant 2$. Denote by

$$S(x_1, x_2, \ldots, x_n) = \sum_{1 \leqslant i < j \leqslant n} d(x_i, x_j)$$

the sum of all pairwise distances between the points from the set $\{x_1, x_2, \ldots, x_n\}$, which we call *total pairwise distance.*

Definition 1. Let $n \geqslant 2$ and let (X, d) be a metric space with $|X| \geqslant n$. We shall say that $T\colon X \to X$ is a *mapping contracting total pairwise distance on n points* if there exists $\alpha \in [0, 1)$ such that the inequality

$$S(Tx_1, Tx_2, \ldots, Tx_n) \leqslant \alpha S(x_1, x_2, \ldots, x_n) \tag{1}$$

holds for all n pairwise distinct points $x_1, x_2, \ldots, x_n \in X$.

Note that the requirement for $x_1, x_2, \ldots, x_n \in X$ to be pairwise distinct is essential, which is confirmed by the following proposition.

Proposition 2. *Suppose that in Definition 1 inequality (1) holds for any n points $x_1, x_2, \ldots, x_n \in X$ with $|\{x_1, x_2, \ldots, x_n\}| = k$, where $2 \leqslant k \leqslant n - 1$. Then T is a mapping contracting total pairwise distance on k points.*

Proposition 3. *Mapping contracting total pairwise distance on m points, $m \geqslant 2$, is a mapping contracting total pairwise distance on n points for all $n > m$.*

Proposition 4. *Mappings contracting total pairwise distance on n points are continuous.*

Let T be a mapping on the metric space X. A point $x \in X$ is called a *periodic point of period* n if $T^n(x) = x$. The least positive integer n for which $T^n(x) = x$ is called the prime period of x. Note that a fixed point is a point of prime period 1.

Theorem 5. *Let $n \geqslant 2$, (X, d) be a complete metric space with $|X| \geqslant n$ and let $T\colon X \to X$ be a mapping contracting total pairwise distance on n points in X. Then T has a periodic point of prime period k, $k \in \{1, \dots, n-1\}$. The number of periodic points is at most $n-1$.*

Let (X, d) be a metric space. Then a mapping $T\colon X \to X$ is called a *contraction mapping* on X if there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leqslant \alpha d(x, y) \tag{2}$$

for all $x, y \in X$.

Corollary 6. *(Banach fixed-point theorem) Let (X, d) be a nonempty complete metric space with a contraction mapping $T\colon X \to X$. Then T admits a unique fixed point.*

The following definition was introduced in [2]. In particular, it is a partial case of Definition 1 when $n = 3$.

Definition 7. Let (X, d) be a metric space with $|X| \geqslant 3$. We shall say that $T\colon X \to X$ is a *mapping contracting perimeters of triangles* on X if there exists $\alpha \in [0, 1)$ such that the inequality

$$d(Tx, Ty) + d(Ty, Tz) + d(Tx, Tz) \leqslant \alpha(d(x, y) + d(y, z) + d(x, z))$$

holds for all three pairwise distinct points $x, y, z \in X$.

The following statement was proved in [2,Theorem 2.4] and it is a direct consequence of Theorem 5 in the case $n = 3$.

Corollary 8. *Let (X, d), $|X| \geqslant 3$, be a complete metric space and let $T\colon X \to X$ be a mapping contracting perimeters of triangles on X. Then T has a fixed point if and only if T does not possess periodic points of prime period 2. The number of fixed points is at most two.*

Proposition 9. *Suppose that under the supposition of Theorem 5 the mapping T has a fixed point x^*, which is a limit of some iteration sequence $x_0, x_1 = Tx_0, x_2 = Tx_1, \dots$ such that $x_i \neq x^*$ for all $i = 1, 2, \dots$. Then x^* is the unique fixed point.*

Recall that for a given metric space X, a point $x \in X$ is said to be an *accumulation point* of X if every open ball centered at x contains infinitely many points of X.

Proposition 10. *Let $n \geqslant 2$, (X, d) be a metric space, $|X| \geqslant n$, and let $T\colon X \to X$ be a mapping contracting total pairwise distance on n points. If x is an accumulation point of X, then inequality (2) holds for all points $y \in X$.*

Corollary 11. *Let $n \geqslant 2$, (X, d) be a metric space, $|X| \geqslant n$, and let $T\colon X \to X$ be a mapping contracting total pairwise distance on n points. If all points of X are accumulation points, then T is a contraction mapping.*

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On the semigroup of non-injective monoid endomorphisms of some extension of the bicyclic monoid

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Let $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ be the semigroup defined in [1] with the two-element family $\mathscr{F}$ of inductive subset of ω. Without loss of generality we may assume that $\mathscr{F}=\{[0,\infty),[1,\infty)\}$.

In the paper [2] we study injective endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ with the two-elements family $\mathscr{F}$ of inductive nonempty subsets of ω. We describe the elements of the semigroup $\boldsymbol{End}_*^1(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ of all injective monoid endomorphisms of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$.

This work is a continuation of [2].

Fix an arbitrary non-negative integer k. For all $i,j\in\omega$ we define transformations γ_k and δ_k of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ in the following way

$$(i,j,[0,\infty))\gamma_k=(i,j,[1,\infty))\gamma_k=(ki,kj,[0,\infty)).$$

$$(i,j,[0,\infty))\delta_k=(ki,kj,[0,\infty)) \qquad \text{and}$$

$$(i,j,[1,\infty))\delta_k=(k(i+1),k(j+1),[0,\infty))$$

By $\boldsymbol{End}_0^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ we denote the semigroup of all non-injective monoid endomorphisms of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ for the family $\mathscr{F}=\{[0,\infty),[1,\infty)\}$.

Theorems 1 and 2 describe the algebraic structure of the semigroup $\boldsymbol{End}_0^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$.

Theorem 1. *If $\mathscr{F}=\{[0,\infty),[1,\infty)\}$, then for any non-injective monoid endomorphism $\mathfrak{e}$ of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ only one of the following conditions holds:*

- *(1) $\mathfrak{e}$ is the annihilating endomorphism, i.e., $\mathfrak{e}=\gamma_0=\delta_0$;*
- *(2) $\mathfrak{e}=\gamma_k$ for some positive integer k;*
- *(3) $\mathfrak{e}=\delta_k$ for some positive integer k.*

Theorem 2. *Let $\mathscr{F}=\{[0,\infty),[1,\infty)\}$. Then for all positive integers k_1 and k_2 the following conditions hold:*

- *(1) $\gamma_{k_1}\gamma_{k_2}=\gamma_{k_1k_2}$;*
- *(2) $\gamma_{k_1}\delta_{k_2}=\gamma_{k_1k_2}$;*
- *(3) $\delta_{k_1}\gamma_{k_2}=\delta_{k_1k_2}$;*
- *(4) $\delta_{k_1}\delta_{k_2}=\delta_{k_1k_2}$.*

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Structure of gradient bifurcations on compact 2-manifolds

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Typical vector fields on compact 2-manifolds are Morse-Smale fields. Among the gradient fields are Morse fields or Morse-Smale gradient-like fields that do not contain closed trajectories. They satisfy three properties:

1) singular points are nondegenerate;
2) there are no saddle connections;
3) α-limit (ω-limit) set of each trajectory is a singular point.

In typical one-parameter field families, one of these conditions is violated. Violation of the first condition leads to a saddle-node bifurcation, and the second to the appearance of a saddle connection. The third condition cannot be violated in gradient fields.

Our main purpose is to describe the global topological structure of typical one-parameter bifurcations of gradient vector fields in the following situations: 1) on closed surfaces in the general situation, 2) on closed surfaces with a minimum number of singular points of the vector field, 3) on surfaces with an edge of a small kind .

To describe a Morse-Smale vector field on a closed surface, we use a cellular structure in which the cells are stable manifolds of singular points. Separatrices are trajectories belonging to one-dimensional stable and unstable manifolds of singular points.

A saddle-node bifurcation can be described as a change in the vector field in which one of the separatrices contracts to a point. If the node is a source, then a pair of cells is reduced in dimensions 0 and 1, and if a node is a sink, then in dimensions 1 and 2.

The saddle connrction bifurcation is described using a subgraph in the form of the letter T, in which one of the three edges is marked.

Chord diagrams are often used to describe the structure of optimal Morse flows (Morse flows with the smallest number of singular points on a given surface). We modify the chord diagram to describe typical gradient bifurcations: 1) for a saddle-node bifurcation using a pair of points on circular arcs, 2) for a saddle-node using a T-insert. In this case, all chords and edges of the T-insert are painted in two colors depending on the alignment of orientations.

Theorem 1. *Optimal saddle-node bifurcations have the same structure if and only if there is an isomorphism of their chord diagrams that preserves the colors of the chords and the location of the two selected points on the circle. For every framed chord diagram there is a corresponding bifurcation.*

Theorem 2. *Optimal saddle connection bifurcations have the same structure if and only if their chord diagrams with a T-insert are isomorphic. For each chord diagram with a T-insert there is a corresponding bifurcation.*

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About square roots of matrices over factorial domains

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Let R be a factorial domain with identity $e \neq 0$. Notations: $\mathrm{R}_{n,m}$ and $\mathrm{R}_{n,m}[\lambda]$ are sets of $(n \times m)$ matrices over the domain R and the polynomial ring $\mathrm{R}[\lambda]$ respectively, 0_n and I_n are the zero and the identity $n \times n$ matrices respectively, $\mathbb{C}$ is the field of complex numbers, $\mathbb{R}$ is the field of real numbers and $\mathbb{Z}$ is the ring of integers.

It is said that an $n \times n$ matrix B is a square root of the matrix $A \in \mathrm{R}_{n,n}$ if $B^2 = A$. The computation of matrix square roots arise in a variety of application domains, including in physics, signal processing, optimal control theory, and many others. The problem of finding square roots from a matrix A over $\mathbb{C}$ or $\mathbb{R}$ is well studied (see [1]–[10] and references therein). Unlike square roots of the complex numbers $\mathbb{C}$, the square root of a matrix over $\mathbb{C}$ may not exist. For example, the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}_{2,2}$ has no square roots. However, for a nonsingular matrix A over $\mathbb{R}$ there always exists a square root over $\mathbb{C}$. We note that the existence of a square root of a matrix A over a field depends on the structure of its elementary divisors corresponding to zero eigenvalues. The structure of square roots over a principal ideal domain of the matrix I_n was written in [14].

It is easy to make sure that for nonsingular matrix $A \in \mathrm{R}_{n,n}$ does not always have a square root over R. For the matrix $A = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix} \in \mathbb{Z}_{2,2}$ there is no square root over $\mathbb{Z}$. However, the matrix $B = \begin{bmatrix} i & 0 \\ -i & i \end{bmatrix}$ over the ring of Gaussian integer $\mathbb{Z}[i]$ is the square root of A. In this report we give conditions under which for a matrix $A \in \mathrm{R}_{n,n}$ there exists a square root over R.

Let R be a factorial domain. For the matrix $A \in \mathrm{R}_{n,n}$ there exists a square root over R if and only if the matrix equation $X^2 = A$ is solvable over R. This equation is solvable if and only if the polynomial matrix $A(\lambda) = I_n\lambda^2 - A$ admits the representation in the form $A(\lambda) = (I_n\lambda - B)(I_n\lambda + B)$, where $B \in \mathrm{R}_{n,n}$. From the last equality we have

$$\det A(\lambda) = a(\lambda) = b(\lambda)\widetilde{b}(\lambda), \tag{1}$$

where $b(\lambda), \widetilde{b}(\lambda) \in \mathrm{R}[\lambda]$ – are monic polynomials of degree n. It is evident that condition (1) is the necessary condition for the existence of a square root of the matrix $A \in \mathrm{R}_{n,n}$.

For the matrix $A \in \mathrm{R}_{n,n}$ and the polynomial $b(\lambda) = \lambda^n + \sum_{i=1}^{n} b_i\lambda^{n-i} \in \mathrm{R}[\lambda]$ (a divisor of the characteristic polynomial of $A(\lambda) = I_n\lambda^2 - A$) we construct the matrices

$$T_A = \begin{bmatrix} \vdots & \vdots \\ O_n & A^3 \\ A^2 & O_n \\ O_n & A^2 \\ A & O_n \\ O_n & A \\ I_n & O_n \\ O_n & I_n \end{bmatrix} \in \mathrm{R}_{n(n+1),2n}, \qquad M_b = \begin{bmatrix} I_n & I_nb_1 & I_nb_2 & \dots & I_nb_{n-1} & I_nb_n \end{bmatrix} \in \mathrm{R}_{n,n(n+1)},$$

$$N_b = \begin{bmatrix} I_nb_1 & I_nb_2 + A & I_nb_3 & \dots & I_nb_{n-1} & I_nb_n & O_n \end{bmatrix} \in \mathrm{R}_{n,(n+1)n}.$$

With matrices T_A, M_b and N_b we associate the $(n \times 2n)$ matrices $M_A = M_bT_A$ and $N_A = N_bT_A$.

In the future, we denote by $d_A(\lambda)$ the g.c.d. minors of $(n-1)$-order of the matrix $A(\lambda)$. By virtue of Theorem 1 in [11], we obtain the following statement.

Proposition 1. *Let a matrix* $B \in \mathrm{R}_{n,n}$ *be a square root of the matrix* $A \in \mathrm{R}_{n,n}$*, i.e.* $B^2 = A$ *and* $\det(I_n\lambda - B) = b(\lambda)$. *If* $\left(b(\lambda), \dfrac{\det A(\lambda)}{b(\lambda)}, d_A(\lambda)\right) = e$*, then the square root* B *is uniquely determined by the characteristic polynomial* $b(\lambda)$ *for the matrix* A.

The proof of the following statements are based on results of papers [12] and [13].

Theorem 2. *Let* $A \in \mathrm{R}_{n,n}$ *and let* $b(\lambda) = \lambda^n + \sum_{i=1}^{n} b_i\lambda^{n-i} \in \mathrm{R}[\lambda]$ *be a divisor of the characteristic polynomial of the matrix* $A(\lambda) = I_n\lambda^2 - A$*, i.e.* $\det A(\lambda) = b(\lambda)\tilde{b}(\lambda)$. *If* $\left(b(\lambda), \tilde{b}(\lambda), d_A(\lambda)\right) = e$*, then for matrix* A *there exists a square root* B *with characteristic polynomial* $b(\lambda) = \det(I_n\lambda - B)$ *if and only if the equation* $XM_A = N_A$ *is solvable. If the square root* B *exists, then matrix* B *is uniquely determined by its characteristic polynomial* $b(\lambda)$.

Corollary 3. *Let* $A \in \mathrm{R}_{n,n}$ *and let* $b(\lambda) = \lambda^n + \sum_{i=1}^{n} b_i\lambda^{n-i} \in \mathrm{R}[\lambda]$ *be a divisor of the characteristic polynomial of the matrix* $A(\lambda) = I_n\lambda^2 - A$*, i.e.* $\det A(\lambda) = b(\lambda)\tilde{b}(\lambda)$. *If* $d_A(\lambda) = \mathrm{const}$*, then for matrix* A *there exists a square root* B *with characteristic polynomial* $b(\lambda) = \det(I_n\lambda - B)$ *if and only if the equation* $XM_A = N_A$ *is solvable. If the square root* B *exists, then matrix* B *is uniquely determined by its characteristic polynomial* $b(\lambda)$.

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Results on boundary behavior of quasiregular and harmonic mappings

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We discuss connections between different conditions involving conformal capacity densities, dilatations and multiplicities of the zeros, and boundary behavior of quasiconformal and related classes of mappings. We compare Carathеódory, Koebe and Lindelöf type results for these classes of mappings to the results from classical function theory as well as those concerning quasiconformal and quasiregular mappings in plane and n-dimensional Euclidean space.

Sufficient conditions for the existence of angular (non-tangential) limit at a boundary point can be obtained, for example, in terms of multiplicities of zeroes of the function, which are required grow fast enough on a given sequence of points approaching the boundary [1, 2, 3] Another condition makes use of makes of capacity density of a non-tangential set at the boundary [4]. We also discuss sharpness of such conditions. This presentation is based on joint work with Daoud Bshouty, Jiaolong Chen, Stavros Evdoridis, Jie Huang, and Matti Vuorinen.

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Einstein Solvmanifolds not based on Nilsolitons

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In this seminar, we describe different techniques to construct pseudo-Riemannian Einstein solvmanifolds, expanding beyond the traditional framework reliant on nilsolitons.

In the first part, we review Einstein solvmanifolds and their construction based on nilsolitons. We will recall the notion of pseudo-Iwasawa and the role of nice nilpotent Lie algebras. Subsequently, we present two different constructions of Einstein solvmanifolds that do not rely on nilsolitons and are peculiar to the indefinite case. The first construction uses contact symplectic reduction (a peculiar feature of pseudo-Sasaki geometry). The second, which is quite new, is based on solving the generalized nilsoliton equation and introduces a new methodology. Both constructions yield examples that are not isometric to any Einstein solvmanifold of pseudo-Iwasawa type.

We will also discuss related geometric structures, such as Sasaki, pseudo-Kähler, and para-Kähler geometries. Time permitting, we will explore open pathways for further research in differential geometry.

This talk is based on joint work with D. Conti and R. Segnan Dalmasso.

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Ricci flow of $G2$-type real flag manifolds

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A real flag manifold is a quotient space $\mathbb{F} = G/P$, where G is a connected Lie group with non-compact real simple associated Lie algebra $\mathfrak{g}$, and P is a parabolic subgroup of G. In [1], ee investigate homogeneous Riemannian geometry on real flag manifolds of the split real form of $\mathbf{g}_2$. We characterize the metrics that are invariant under the action of a maximal compact subgroup of G2 and we explore the Ricci flow for the case where the isotropy representation has no equivalent summands, employing techniques from the qualitative theory of dynamical systems. This is joint work with Brian Grajales and Gabriel Rondon.

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Approximation by interpolation trigonometric polynomials on the sets of infinitely differentiable functions

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Denote by $C^{\psi}_{\beta}L_1$ the set of 2π–periodic functions, which for all $x\in\mathbb{R}$ can be represented as convolutions of the form

$$f(x)=\frac{a_0}{2}+\frac{1}{\pi}\int\limits_{-\pi}^{\pi}\Psi_{\beta}(x-t)\varphi(t)dt,\ a_0\in\mathbb{R},\ \varphi\in L_1,\ \varphi\perp 1 \tag{1}$$

with the generating kernel Ψ_{β} of the form

$$\Psi_{\beta}(t)=\sum_{k=1}^{\infty}\psi(k)\cos\big(kt-\frac{\beta\pi}{2}\big),\ \psi(k)>0,\ \beta\in\mathbb{R},$$

such that

$$\sum_{k=1}^{\infty}\psi(k)<\infty.$$

The function φ in equality (1) is called as (ψ,β)–derivative of the function f and is denoted by f^{ψ}_{β} $(\varphi(\cdot)=f^{\psi}_{\beta}(\cdot))$ [1].

We study approximation properties of the sets $C^{\psi}_{\beta}L_1$, where we use as approximation aggregate the classical interpolation trigonometric Lagrange polynomials, which are defined by odd number of uniformly distributed interpolation nodes.

For arbitrary function $f(x)$ from C by $\widetilde{S}_{n-1}(f;x)$, $n\in\mathbb{N}$, we will denote the trigonometric polynomial of the order $n-1$, which interpolates $f(x)$ in the nodes $x_k^{(n-1)}=\frac{2k\pi}{2n-1}$, $k\in\mathbb{Z}$, namely, such that

$$\tilde{S}_{n-1}(f;x_k^{(n-1)})=f(x_k^{(n-1)}),\ k=0,1,...,2n-2.$$

The polynomial $\widetilde{S}_{n-1}(f;\cdot)$ is unequivocally defined by mentioned interpolation conditions, is called as Lagrange interpolation polynomial and can be represented in the explicit form via Dirichlet kernel

$$D_{n-1}(t)=\frac{1}{2}+\sum_{k=1}^{n-1}\cos kt=\frac{\sin(n-\frac{1}{2})t}{2\sin\frac{t}{2}}$$

in the following way

$$\tilde{S}_{n-1}(f;x)=\frac{2}{2n-1}\sum_{k=0}^{2n-2}f(x_k^{(n-1)})D_{n-1}(x-x_k^{(n-1)}).$$

Let $\mathcal{T}_{2n-1}$ be the space of all trigonometric polynomials of degree at most $n-1$ and let $E_n(f)_{L_1}$ be the best approximation of the function $f\in L_1$ in the metric of space L_1, by the trigonometric polynomials t_{n-1} of degree $n-1$, i.e.,

$$E_n(f)_{L_1} = \inf_{t_{n-1}\in\tau_{2n-1}} \|f - t_{n-1}\|_{L_1}.$$

Denote by $\tilde{\rho}_n(f;\cdot)$ the deviation of the function $f \in C$ from its interpolation Lagrange polynomial $\tilde{S}_{n-1}(f;\cdot)$

$$\tilde{\rho}_n(f;x) = f(x) - \tilde{S}_{n-1}(f;x).$$

Let

$$\mathfrak{M}=\left\{\psi\in C[1,\infty)\colon \psi(t)>0, \psi(t_1 - 2\psi((t_1+t_2)/2) + \psi(t_2) \geq 0 \ \forall t_1, t_2 \in [1,\infty), \lim_{t\to\infty}\psi(t)=0\right\}.$$

We consider for each function $\psi \in \mathfrak{M}$ the following characteristics

$$\alpha(t) = \alpha(\psi;t) := \frac{\psi(t)}{t|\psi'(t)|}, \quad \psi'(t) := \psi'(t+0),$$

$$\lambda(t) = \lambda(\psi;t) := \frac{\psi(t)}{|\psi'(t)|}.$$

As it was shown in [2], if $\alpha(t) \downarrow 0$ as $t \to \infty$, then the sets $C^{\psi}_{\beta}L_1$ are the sets of infinitely differentiable functions.

Our aim is to establish the asymptotically best possible interpolation analogues of the Lebesgue type inequalities for the functions f from the sets $C^{\psi}_{\beta}L_1$, $\beta \in \mathbb{R}$, where the upper estimates of the quantities $|\tilde{\rho}_n(f;x)|$, $x \in \mathbb{R}$, are expressed via the best approximations $E_n(f^{\psi}_{\beta})_{L_1}$.

The following theorem takes place.

Theorem 1. *Let $\psi \in \mathfrak{M}$ and characteristics $\alpha(t)$ and $\lambda(t)$ satisfy the conditions*

$$\alpha(t) \downarrow 0, \quad t \to \infty,$$

$$\lambda(t) \uparrow \infty, \quad t \to \infty.$$

Then, for arbitrary function $f \in C^{\psi}_{\beta}L_1$, $\beta \in \mathbb{R}$, in every point $x \in \mathbb{R}$ for all $n \in \mathbb{N}$ such that

$$\alpha(n) \leq \frac{1}{4},$$

the following inequality takes place

$$|\tilde{\rho}_n(f;x)| \leq \frac{2}{\pi}\left|\sin\frac{2n-1}{2}x\right| \psi(n)\lambda(n)\left(1 + 4\alpha(n) + \frac{1}{\lambda(n)}\right) E_n(f^{\psi}_{\beta})_{L_1}.$$

Moreover for arbitrary function $f \in C^{\psi}_{\beta}L_1$ one can find the function $\mathcal{F}(\cdot) = \mathcal{F}(f;n;x,\cdot)$ such that $E_n(\mathcal{F}^{\psi}_{\beta})_{L_1} = E_n(f^{\psi}_{\beta})_{L_1}$ and the following equality takes place

$$|\tilde{\rho}_n(\mathcal{F};x)| = \frac{2}{\pi}\left|\sin\frac{2n-1}{2}x\right| \psi(n)\lambda(n)\left(1 + \xi_1\alpha(n) + \frac{\xi_2}{\lambda(n)}\right) E_n(f^{\psi}_{\beta})_{L_1},$$

where $-4(1+2\pi) \leq \xi_1 < \frac{8}{3}(1+\pi)$, $-(1+2\pi) \leq \xi_2 \leq 2(1+\pi)$.

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Fundamental solution of non-Archimedean pseudo-differential equation of p-adic argument

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The Vladimirov-Taibleson pseudo-differential operator D^α plays a role of a differential operator in the p-adic analysis (see [1, 3]). The analogue of the wave equation for radial functions in t on non-Archimedean spaces

$$D_t^\alpha u - D_x^{\alpha,n} u = 0 \tag{1}$$

was studied in [2].

In present work the fundamental solution of a more general Cauchy problem for the functions of two p-adic variables, radial in t, was found. The main result is stated in Theorem 2. Theorem 3 proves the uniqueness of the solution in the Lizorkin space of locally constant functions $\Phi(\mathbb{Q}_p^n)$, and Theorem 4 gives an estimate of the norm of the solution in L^1-space.

Let $0 < \alpha < 1$, $\beta > 0$. Consider the eigenvalue problem

$$D^\alpha u = \lambda u,\ \lambda = p^{\beta N},\ N \in \mathbb{Z}, \tag{2}$$

where u is not identically zero.

We also suppose that

$$\beta = K\alpha \text{ for some } K \in \mathbb{N}. \tag{3}$$

Proposition 1. *If the condition (3) holds, the equation (2) has the set of solutions in $\Phi(\mathbb{Q}_p)$ of the following form for $N \in \mathbb{Z}$:*

$$u_N(t) = \begin{cases} C_N p^{KN}(1 - \frac{1}{p}), & |t|_p \le p^{-KN}; \\ -C_N p^{KN-1}, & |t|_p = p^{-KN+1}; \\ 0, & |t|_p \ge p^{-KN+2}. \end{cases} \tag{4}$$

Let $0 < \alpha < 1$, $\beta > 0$. We consider the Cauchy problem

$$D^\alpha_{|t|_p} u(|t|_p, x) - D^\beta_x u(|t|_p, x) = 0,\ (t,x) \in \mathbb{Q}_p^+ \times \mathbb{Q}_p^n, \tag{5}$$

$$u(0,x) = u_0(x),\ x \in \mathbb{Q}_p^n, \tag{6}$$

where $u = u(|t|_p, x)$ is a radial function with respect to t, $n \ge 1$.

Theorem 2. *Let $0 < \alpha < 1$, $\beta > 0$ such that the condition (3) holds. Suppose that the function u_0 is in $\Phi(\mathbb{Q}_p)^n$. Then the Cauchy problem (5)-(6) has a solution $u = u(|t|_p, x)$, radial in t, that belongs to the space $\Phi(\mathbb{Q}_p^+)$ for each $x \in \mathbb{Q}_p$, and belongs to $\Phi(\mathbb{Q}_p^n)$ for each $t \in \mathbb{Q}_p^+$.*

If the condition (3) does not hold, then the equation (5) has only a zero solution $u(t,x) \equiv 0$, $t, x \in \mathbb{Q}_p^+ \times \mathbb{Q}_p$.

The solution built in the proof of Theorem 2 has the following form

$$u(|t|_p, x) = (\mathcal{F}^{-1}_{\xi\to x}\hat{u})(|t|_p, x) = \left((\mathcal{F}^{-1}_{\xi\to x}b) * u_0\right)(|t|_p, x),\ (t, x) \in \mathbb{Q}_p^+ \times \mathbb{Q}_p^n, \tag{7}$$

We consider the problem (5)-(6) in the class of generalized functions, radial in t.

Denote by $\Phi'(\mathbb{Q}_p^+, \Phi'(\mathbb{Q}_p^n))$ the set of distributions over the test function space $\Phi(\mathbb{Q}_p^n)$, with values in $\Phi'(\mathbb{Q}_p^n)$.

Theorem 3. *Let $F \in \Phi'(\mathbb{Q}_p^+, \Phi'(\mathbb{Q}_p^n))$ be a generalised solution of the equation (5), that is*

$$\langle\langle F, D_t^\alpha \varphi_1\rangle, \varphi_2\rangle = \langle\langle F, \varphi_1\rangle, D_x^\beta \varphi_2\rangle,$$

for any $\varphi_1 \in \Phi(\mathbb{Q}_p^+)$, $\varphi_2 \in \mathbb{Q}_p$. If F is radial in t, then $F \in \mathcal{D}(\mathbb{Q}_p^+, \Phi'(\mathbb{Q}_p^n))$. If, in addition, $F(0, x) = 0$, then $F(t, x) \equiv 0$.

It follows from Theorem 3 that the solutions of the Cauchy problems constructed in Theorem 2 are unique in the class of radial in t, bounded locally constant functions.

Theorem 4. *Suppose that the conditions of Theorem 2 hold. Then the solution of the problem (5)-(6), defined in (7), satisfies the following estimate in $L^1(\mathbb{Q}_p^n)$ in variable x*

$$||u(|t|_p, \cdot)||_{L_1(\mathbb{Q}_p^n)} \leq p^{2n\gamma}||u_0||_{L_1(\mathbb{Q}_p^n)}, \tag{8}$$

where $\gamma \geq \dfrac{2}{K}$ is a positive constant.

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On the semigroup of injective monoid endomorphisms of a some extension of the bicyclic semigroup

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In this paper we shall follow the semigroup terminology of [5].

By ω we denote the set of all non-negative integers.

Let $\mathscr{P}(\omega)$ be the family of all subsets of ω. For any $F \in \mathscr{P}(\omega)$ and any integer n we put $n + F = \{n + k\colon k \in F\}$ if $F \neq \varnothing$ and $n + \varnothing = \varnothing$. A subfamily $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is called *ω-closed* if $F_1 \cap (-n + F_2) \in \mathscr{F}$ for all $n \in \omega$ and $F_1, F_2 \in \mathscr{F}$. For any $a \in \omega$ we denote $[a) = \{x \in \omega\colon x \geqslant a\}$.

On the set $\boldsymbol{B}_\omega = \omega \times \omega$ we define the semigroup operation "$\cdot$" in the following way

$$(i_1, j_1) \cdot (i_2, j_2) = \begin{cases} (i_1 - j_1 + i_2, j_2), & \text{if } j_1 \leqslant i_2; \\ (i_1, j_1 - i_2 + j_2), & \text{if } j_1 \geqslant i_2. \end{cases}$$

It is well known that the bicyclic monoid is isomorphic to the semigroup $\boldsymbol{B}_\omega$.

The following construction is introduced in [1].

Let $\mathscr{F}$ be an ω-closed subfamily of $\mathscr{P}(\omega)$. On the set $\boldsymbol{B}_\omega \times \mathscr{F}$ we define the semigroup operation "$\cdot$" in the following way

$$(i_1, j_1, F_1) \cdot (i_2, j_2, F_2) = \begin{cases} (i_1 - j_1 + i_2, j_2, (j_1 - i_2 + F_1) \cap F_2), & \text{if } j_1 \leqslant i_2; \\ (i_1, j_1 - i_2 + j_2, F_1 \cap (i_2 - j_1 + F_2)), & \text{if } j_1 \geqslant i_2. \end{cases}$$

In [1] is proved that if the family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is ω-closed then $(\boldsymbol{B}_\omega \times \mathscr{F}, \cdot)$ is a semigroup. Moreover, if an ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ contains the empty set $\varnothing$ then the set $\boldsymbol{I} = \{(i, j, \varnothing) \colon i, j \in \omega\}$ is an ideal of the semigroup $(\boldsymbol{B}_\omega \times \mathscr{F}, \cdot)$. For any ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ the following semigroup

$$\boldsymbol{B}_\omega^{\mathscr{F}} = \begin{cases} (\boldsymbol{B}_\omega \times \mathscr{F}, \cdot)/\boldsymbol{I}, & \text{if } \varnothing \in \mathscr{F}; \\ (\boldsymbol{B}_\omega \times \mathscr{F}, \cdot), & \text{if } \varnothing \notin \mathscr{F} \end{cases}$$

is defined in [1].

In the paper [2] injective endomorphisms of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ with the two-elements family $\mathscr{F}$ of inductive nonempty subsets of ω are studies. Here the authors describe the elements of the semigroup $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ of all injective monoid endomorphisms of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$, and show that Green's relations $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, $\mathscr{D}$, and $\mathscr{J}$ on $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ coincide with the relation of equality. In [3, 4] the semigroup $\boldsymbol{End}^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ of all monoid endomorphisms of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$ is studied.

Example 1. Let $\mathscr{F}^3 = \{[0), [1), [2)\}$. Fix an arbitrary positive integer k. We define the transformation $\alpha_{[k]}$ of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}^3}$ in the following way

$$(i, j, [p))\alpha_{[k]} = \begin{cases} (ki, kj, [p)), & \text{if } p \in \{0, 1\}; \\ (k(i+1) - 1, k(j+1) - 1, [2)), & \text{if } p = 2, \end{cases}$$

for all $i, j \in \omega$. It is obvious that $\alpha_{[k]}$ is an injective transformation of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}^3}$.

Lemma 2. *For an arbitrary positive integer k the transformation $\alpha_{[k]} \colon \boldsymbol{B}_\omega^{\mathscr{F}^3} \to \boldsymbol{B}_\omega^{\mathscr{F}^3}$ is an injective monoid endomorphism of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}^3}$.*

Theorem 3. *Let $\mathscr{F}^3 = \{[0), [1), [2)\}$ and ε be an injective monoid endomorphism of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}^3}$. Then $\varepsilon = \alpha_{[k]}$ for some positive integer k.*

By $(\mathbb{N}, \cdot)$ we denote the multiplicative semigroup of positive integers.

Theorem 4. *Let $\mathscr{F}^3 = \{[0), [1), [2)\}$. Then the monoid $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}^3})$ of all injective endomorphisms of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}^3}$ is isomorphic to $(\mathbb{N}, \cdot)$.*

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On some nonlocal critical equations

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Aim of this talk will be to discuss some existence and multiplicity results for critical nonlocal fractional problems got via variational and topological methods. In particular we will present recent contributions got in the joint paper [1].

Fractional and nonlocal operators appear in various models coming from many different fields. This is one of the reason why, recently, nonlocal fractional problems attracted the interest of the entire scientific community and not just the mathematical one.

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On boundary estimates of mappings, acting onto domains with a locally quasiconformal boundary

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The following definitions are from [1]. A path γ in $\mathbb{R}^n$ is a continuous mapping $\gamma : \Delta \to \mathbb{R}^n$ where Δ is an interval in $\mathbb{R}$. Its locus $\gamma(\Delta)$ is denoted by $|\gamma|$. Given a family Γ of paths γ in $\mathbb{R}^n$, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for Γ, abbr. $\rho \in \operatorname{adm}\Gamma$, if $\int\limits_{\gamma} \rho(x)|dx| \geqslant 1$ for each (locally rectifiable) $\gamma \in \Gamma$. The *modulus* of Γ is defined by the relation

$$M(\Gamma) := \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^n(x)\,dm(x) \qquad (1)$$

interpreted as $+\infty$ if $\operatorname{adm}\Gamma = \varnothing$. Everywhere below, unless otherwise stated, the boundary and the closure of a set are understood in the sense of the extended Euclidean space $\overline{\mathbb{R}^n}$.

Let $Q : \mathbb{R}^n \to [0, \infty]$ be Lebesgue measurable function. We will say that f *satisfies the inverse Poletsky's inequality* if the ratio

$$M(\Gamma) \leqslant \int\limits_{f(D)} Q(y) \cdot \rho_*^n(y)\, dm(y) \tag{2}$$

holds for any family of (locally rectifiable) paths Γ in D and for any $\rho_* \in \operatorname{adm} f(\Gamma)$. Note that estimates of the type (2) are well known and holds for classes of mappings (see, e.g., [2, Theorem 6.7.II] and [3, theorem 8.5]).

Given sets E and F and a given domain D in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, we denote by $\Gamma(E, F, D)$ the family of all paths $\gamma : [0, 1] \to \overline{\mathbb{R}^n}$ joining E and F in D, that is, $\gamma(0) \in E$, $\gamma(1) \in F$ and $\gamma(t) \in D$ for all $t \in (0, 1)$. In accordance with [4], a domain D in $\mathbb{R}^n$ is called *quasiextremal distance domain (QED-domain for short)* if

$$M(\Gamma(E, F, \mathbb{R}^n)) \leqslant A_0 \cdot M(\Gamma(E, F, D)) \tag{3}$$

for some finite number $A_0 \geqslant 1$ and all continua E and F in D. In the extended Euclidean space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ we use the *spherical (chordal) metric* $h(x, y) = |\pi(x) - \pi(y)|$, where π is a stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2}) = \{x \in \mathbb{R}^{n+1} : |x - e_{n+1}/2| = 1/2\}$ in $\mathbb{R}^{n+1}$, and

$$h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}\,,$$

$$h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}}\,, \quad x \neq \infty \neq y \tag{4}$$

(see e.g. [1, Definition 12.1]). In what follows, given $A, B \subset \overline{\mathbb{R}^n}$ we set $h(A, B) = \inf\limits_{x \in A, y \in B} h(x, y)$, where h is a chordal metric in (4). Consider the following definition that has been proposed by Näkki [5], cf. [6]. The boundary of a domain D is called *locally quasiconformal* if every point $x_0 \in \partial D$ has a neighborhood U, for which there exists a quasiconformal mapping φ of U onto the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ such that $\varphi(\partial D \cap U)$ is the intersection of the unit sphere $\mathbb{B}^n$ with a coordinate hyperplane $x_n = 0$, where $x = (x_1, \ldots, x_n)$. Note that, with slight differences in the definition, domains with such boundaries are also called collared domains.

Given $\delta > 0$, domains $D, D' \subset \mathbb{R}^n$, $n \geqslant 2$, a nondegenerate continuum $A \subset D'$ and a Lebesgue-measurable function $Q : D' \to [0, \infty]$ denote by $\mathfrak{S}_{\delta, A, Q}(D, D')$ the family of all open discrete and closed mappings f of the domain D onto the domain D' satisfying the condition (2) and such that $h(f^{-1}(A), \partial D) \geqslant \delta$. The following statement is true.

Theorem 1. *Let $Q \in L^1(D')$, let D be a QED-domain, and D' is a bounded domain with a locally quasiconformal boundary. Then any mapping $f \in \mathfrak{S}_{\delta, A, Q}(D, D')$ which satisfies the relation (2) has a continuous extension $f : \overline{D} \to \overline{D'}$, while, for each point $x_0 \in \partial D$ there will be U neighborhoods of this point and constants $C = C(n, A, D, D', x_0) > 0$ and $0 < \alpha = \alpha(n, A, D, D', x_0) \leqslant 1$ such that*

$$|\overline{f}(x) - \overline{f}(y)|^{\frac{n}{\alpha^2}} \leqslant \frac{C \cdot \|Q\|_1}{\log\left(1 + \frac{\delta}{2|x - y|}\right)} \tag{5}$$

for all $x, y \in U \cap \overline{D}$, where $\|Q\|_1$ is the norm of the function Q in $L^1(D')$.

The result mentioned above is accepted for publication in [7].

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Some properties of affine ruled submanifolds

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We consider an affine ruled submanifolds of arbitrary dimension and codimension in the classical sense, that is, a ruled submanifolds over a curve.

Using the equiaffine theory of curves in an arbitrary affine space [1] we choose the most convenient parameterization and transversal distribution of the affine immersion of a ruled submanifold in general case, i. e., of arbitrary dimension and codimension.

All affine characteristics (induced connection, transversal connection, affine fundamental forms, Weingarten operators, curvature tensor) of such an affine immersion are found depending on the characteristics of the base curve and rectilinear generators.

We find the conditions for a base curve and directions of rectilinear generators so that the induced connection is flat. These conditions coincide with the already known properties of affine immersions with flat connection ([2]-[6]). Also we find the conditions for a base curve and directions of rectilinear generators so that the chosen transversal distribution is equiaffine.

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On some vanishing theorems of global character about geodesic mappings of complete Riemannian spaces

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Generalization of Bocher technique (see for example, [1]) allows to broad to the complete Riemannian spaces a lot of theorems of geodesic unique definiteness on the whole proved previousely only for the compact ones (see for example, [2]). It seems to be interesting to indicate some of them.

Theorem 1. *Complete Ricci semi-symmetric Riemannian* C^r*-spaces* V^n $(n > 2,\ r > 4)$ *with positively definite metric form, Einstein tensor of which doesn't equal to zero, don't admit non-trivial (different from the affine) geodesic mappings on the whole.*

Theorem 2. *Complete Riemannian* C^r*-spaces* V^n $(n > 2,\ r > 4)$ *with positively definite metric form and non-negative scalar curvature* $(R \geq 0)$ *don't admit non-trivial (different from the affine) geodesic mappings on the whole.*

Theorem 3. *Complete Ricci semi-symmetric Riemannian* C^r*-spaces* V^n $(n > 2,\ r > 4)$ *with positively definite Ricci form, Einstein tensor of which doesn't equal to zero, scalar curvature of which preserves its sign* $(R \geq 0$ *or* $R \leq 0$ *everywhere in* $V^n)$ *don't admit non-trivial (different from the affine) geodesic mappings on the whole.*

Theorem 4. *Complete Ricci semi-symmetric Riemannian* C^r*-spaces* V^n $(n > 2,\ r > 4)$ *with positively definite Einstein form don't admit non-trivial (different from the affine) geodesic mappings on the whole.*

Examples of Riemannian spaces of the considered types are known.

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Subwreath product as structure of normal subgroups of permutational wreath products

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In this research we continue our previous investigation of wreath product normal structure [1, 2] Normal subgroups and there structures for finite and infinite iterated wreath products $S_{n_1} \wr \ldots \wr S_{n_m}$, $n, m \in \mathbb{N}$ and $A_n \wr S_n$ are founded.

Let $k(\pi)$ be the number of cycles in decomposition of permutation π of degree n.

The number $n - k(\pi)$ is denoted by $dec(\pi)$, and is called a decrement [6] of permutation π. As well known [6] the minimal number of transpositions in factorization of a permutation π on transpositions is happen to be equal to $dec(\pi)$. We set $dec(e) = 0$. If $\pi_1, \pi_2 \in S_n$, then the following formula holds:

$$dec(\pi_1 \cdot \pi_2) = dec(\pi_1) + dec(\pi_2) - 2m, m \in \mathbf{N}, \tag{1}$$

Definition 1. The permutational *subwreath product* $G \wr\wr H$ is the semi-direct product $G \ltimes \tilde{H}^X$, where G acts on the subdirect product [4] $\tilde{H}^X$ by the respective permutations of the subdirect factors. Provided the specification of $\tilde{H}^X$ is established separately.

Definition 2. The set of elements from $S_n \wr S_n, n \geqslant 3$ which presented by the tableaux of form: $[e]_0,\ [a_1, a_2, \ldots, a_n]_1$, satisfying the following condition

$$\sum_{i=1}^{n} dec([a_i]_1) = 2k, k \in \mathbb{N}, \tag{2}$$

be called set of type $\widetilde{A}_n^{(1)}$. Note that condition (2) uniquely identifies subdirect product.

The set $\widetilde{A}_n^{(1)}$ is normal subgroup having **normal rank** 2 [7] in $S_n \wr S_n$ and be denoted by $E \wr\wr \widetilde{A}_n$. We spread this definition on 3-multiple wreath product by recursive way.

Definition 3. *The subgroup* $E \wr \widetilde{A}_n^{(1)}$ *be denoted by* $\widetilde{A}_n^{(2)}$.

Furthermore we prove that $E \wr \widetilde{A}_n^{(2)} \lhd S_n \wr S_n \wr S_n$. The order of $E \wr \widetilde{A}_n^{(2)}$ is $(n!)^{3n} : 2^3$. The subgroup $\widetilde{A}_n^{(1)}$ has **normal rank** 2 in $S_n \wr S_n$.

Definition 4. The set of elements from $S_n \wr S_n \wr S_n, n \geqslant 3$ presented by the tables [3] form: $[e]_1,\ [e, e, \ldots, e]_2, [a_1, a_2, \ldots, a_n]_2$, satisfying the following condition

$$\sum_{i=1}^{n} dec([a_i]_2) = 2k, k \in \mathbb{N}, \tag{3}$$

be denoted by $\widetilde{A}_{n^2}^{(2)}$. Note that condition (3) uniquely identifies subdirect product in $\prod_{i=1}^{n^2} S_n$ as base of subwreath product, the similar subdirect product describing commutator of wreath product was investigated by us in [8] in research of pronormality it appears in [9].

Proposition 5. *The subgroup* $\widetilde{A}_n^{(1)} \lhd S_n \wr S_n$ *as well as* $\widetilde{A}_n^{(2)} \lhd S_n \wr S_n \wr S_n$. *Furthermore* $\widetilde{A}_n^{(2)} \lhd \widetilde{A}_{n^2}^{(2)}$.

Definition 6. *A subgroup in* $S_n \wr S_n$ *is called* $\widetilde{T_n}$ *if it consists of:*

(1) elements of $E \wr A_n$ *,*

(2) elements with the tableau [3] *presentation* $[e]_1,\ [\pi_1, \ldots, \pi_n]_2$, *that* $\pi_i \in S_n \setminus A_n$.

One easy can validates a correctness of this definition, i.e. that the set of such elements form a subgroup and its normality. This subgroup has structure

$$\tilde{T}_n \simeq (\underbrace{A_n \times A_n \times \cdots \times A_n}_{n}) \rtimes C_2 \simeq \underbrace{S_n \boxplus S_n \ldots \boxplus S_n}_{n},$$

where the operation $\boxplus$ *of a subdirect product is subject of item 1) and 2)*

Remark 7. The order of $\widetilde{T_n}$ is $\frac{(n!)^n}{2^{n-1}}$.

Definition 8. The unique minimal normal subgroup is called the monolith.

Theorem 9. *The monolith of* $S_n \wr S_m$ *is* $e \wr A_m$.

Theorem 10. *Proper normal subgroups in* $S_n \wr S_m$*, where* $n, m \geq 3$ *with* $n, m \neq 4$ *are of the following types:*

(1) subgroups that act only on the second level are

$$E \wr\wr \widetilde{A_m},\ \widetilde{T_m},\ E \wr S_m, E \wr A_m,$$

(2) subgroups that act on both levels are $A_n \wr\wr \widetilde{A_m}$*,* $S_n \wr\wr \widetilde{A_m}$*,* $A_n \wr S_m$*,*

wherein the subgroup $S_n \wr\wr \widetilde{A_m} \simeq S_n \leftthreetimes (\underbrace{S_m \boxtimes S_m \boxtimes S_m \ldots \boxtimes S_m}_{n})$ *endowed with the subdirect product satisfying to condition* (2).

Theorem 11. *The full list of normal subgroups of* $W = S_n \wr S_n \wr S_n$ *consists of 50 normal subgroups. These subgroups are the following:*

1. ***Type*** T_{023} *contains:* $E \wr\wr \tilde{A}_n \wr\wr H$, $\widetilde{T_n} \wr H$*, where* $H \in \{\tilde{A}_n, \tilde{A}_{n^2}, S_n\}$*. There are 6 subgroups.*
2. ***The second type of subgroups is subclass in*** T_{023} *with new base of wreath product subgroup* $\tilde{A}_{n^2}$*:* $E \wr S_n \wr\wr \tilde{A}_{n^2}$*,* $E \wr N_i(S_n \wr S_n)$*. Therefore this class has 12 new subgroups. Thus, the total number of normal subgroups in* ***Type*** T_{023} *is 18.*
3. ***Type*** T_{003}*:* $A^{(3)}_{00(n^2)}$*,* $\widetilde{T_{n^2}}$*,* $\widetilde{T_n}^{(3)}$*.*
4. ***Type*** T_{123}*:* $N_i(S_n \wr S_n) \wr S_n$*,* $N_i(S_n \wr S_n) \wr \tilde{A}_n$ *and* $N_i(S_n \wr S_n) \wr \tilde{A}_{n^2}$*. Thus, there are 29 new normal subgroups in* T_{123}*, taking into account a repetition.*

Remark 12. Note that $E \wr \widetilde{A}_n^{(1)} \simeq E \wr (E \wr\wr \widetilde{A}_n)$ contains in the family $E \wr N_i(S_n \wr S_n)$.

We denote by $Aut_f X^*$ the group of all finite automorphism of spherically homogeneous rooted tree.

Theorem 13. *Let* $H \lhd Aut_f X^*$ *having depth* k*, then* H *contains* k*-th level subgroup* P *having all even vertex permutations* $p_{ki} \in A_n$ *on* X^k *and trivial permutations in vertices of rest of levels.*

Furthermore P *is normal in* $Aut_f X^*$ *provided* k *is last active level of* $Aut_f X^*$*.*

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Uniform approximation by Fourier sums in Weyl–Nagy classes $W^r_{\beta,1}$

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We investigate asymptotic behavior of the least upper bounds of approximations in the uniform metric by Fourier sums $S_{n-1}(f;\cdot)$ of classes $W^r_{\beta,1}$ of 2π-periodic Weyl–Nagy differentiable functions f.

Let L_p, $1 \le p \le \infty$, and C be the spaces of 2π–periodic functions with standard norms $\|\cdot\|_{L_p}$ and $\|\cdot\|_C$, respectively.

Further, let $W^r_{\beta,p}$, $r > 0$, $\beta \in \mathbb{R}$, $1 \le p \le \infty$, be classes of 2π-periodic functions f that can be represented in the form of convolution

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int\limits_{-\pi}^{\pi} \varphi(x-t) B_{r,\beta}(t)dt, \quad a_0 \in \mathbb{R}, \tag{1}$$

with Weyl–Nagy kernels of the form $B_{r,\beta}(t) = \sum\limits_{k=1}^{\infty} k^{-r} \cos\left(kt - \frac{\beta\pi}{2}\right)$, of function φ satisfying the condition $\varphi \in B^0_p = \left\{\varphi \in L_p : \|\varphi\|_p \le 1, \int\limits_{-\pi}^{\pi} \varphi(t)dt = 0\right\}$.

The classes $W^r_{\beta,p}$ are called the Weyl–Nagy classes and the function φ in representation (1) is called the (r,β)-derivative of the function f in the Weyl–Nagy sense and denoted by f^r_β.

Theorem 1. *Let* $r > 2$, $\beta \in \mathbb{R}$, *and* $n \in \mathbb{N}$. *The following estimate is true*

$$\mathcal{E}_n(W^r_{\beta,1})_C = \sup_{f \in W^r_{\beta,1}} \|f(\cdot) - S_{n-1}(f;\cdot)\|_C = \frac{1}{n^r}\left(\frac{1}{\pi(1-e^{-r/n})} + \mathcal{O}(1)\delta_{r,n}\right), \tag{2}$$

where $\mathcal{O}(1)$ *is a quantity uniformly bounded in all analyzed parameters,*

$$\delta_{r,n} = \begin{cases} 1 + \dfrac{n}{r(r-2)}, & 2 < r \le n+1, \\ \dfrac{r}{n^2} e^{-r/n}, & n+1 \le r \le n^2, \\ e^{-r/n} & r \ge n^2. \end{cases}$$

Remark 2. Estimate (2) was published for $r \ge \sqrt{n} + 1$ in [1, 2] (2019, 2022).

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On the asymptotic behavior of solutions to nonlinear Beltrami equation

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Let $\mathbb{C}$ be the complex plane. In the complex notation $f = u + iv$ and $z = x + iy$, the *Beltrami equation* in a domain $G \subset \mathbb{C}$ has the form

$$f_{\overline{z}} = \mu(z) f_z, \tag{1}$$

where $\mu\colon G \to \mathbb{C}$ is a measurable function and

$$f_{\overline{z}} = \frac{1}{2}(f_x + i f_y) \qquad \text{and} \qquad f_z = \frac{1}{2}(f_x - i f_y)$$

are formal derivatives of f in $\overline{z}$ and z, while f_x and f_y are partial derivatives of f in the variables x and y, respectively.

We consider the following equation written in the polar coordinates (r, θ)

$$f_\theta = \sigma(re^{i\theta})\,|f_r|^m\, f_r. \tag{2}$$

We rewrite the equation (2) in the Cartesian form:

$$f_{\overline{z}} = \frac{z}{\overline{z}}\,\frac{1 + i\sigma(z)\,|z|^{-m-1}|zf_z + \overline{z}f_{\overline{z}}|^m}{1 - i\sigma(z)\,|z|^{-m-1}|zf_z + \overline{z}f_{\overline{z}}|^m}\, f_z. \tag{3}$$

Assuming $m = 0$, the equation (3) also becomes the standard linear Beltrami equation (1) with

$$\mu(z) = \frac{z}{\overline{z}}\,\frac{1 + i\sigma(z)/|z|}{1 - i\sigma(z)/|z|}\ .$$

Choosing $m = 0$ and $\sigma = i|z|$ in (3), we arrive at the classical Cauchy-Riemann system. Later on we assume that $m > 0$.

A mapping $f\colon G \to \mathbb{C}$ is called *regular at a point* $z_0 \in G$, if f has the total differential at this point and its Jacobian $J_f = |f_z|^2 - |f_{\overline{z}}|^2$ does not vanish. A homeomorphism f of Sobolev class $W^{1,1}_{\mathrm{loc}}$ is called *regular*, if $J_f > 0$ a.e. By a *regular solution of the equation* (3) we call a regular homeomorphism $f\colon G \to \mathbb{C}$, which satisfies (3) a.e. in G.

Later on, we use the following notations:

$$B_r = \{z \in \mathbb{C} : |z| < r\}\,, \quad \mathbb{B} = \{z \in \mathbb{C} : |z| < 1\}$$

and

$$\gamma_r = \{z \in \mathbb{C} : |z| = r\}.$$

Theorem 1. *Let $f:\mathbb{B}\to\mathbb{C}$ be a regular homeomorphic solution of the equation (3) which belongs to Sobolev class $W^{1,2}_{\rm loc}$, and normalized by $f(0)=0$. Assume that $C>0$ and the coefficient $\sigma:\mathbb{B}\to\mathbb{C}$ satisfies the following condition*

$$\int\limits_{\gamma_r}\frac{|\sigma(z)|^{m+2}}{(\operatorname{Im}\sigma(z))^{m+1}}\,|dz|\leqslant C\,r^2$$

for a.a. $r\in(0,1)$. Then

$$\limsup_{z\to0}\frac{|f(z)|}{|z|}\geqslant\left(\frac{2\pi}{C}\right)^{\frac{1}{m}}.$$

Corollary 2. *Let $f:\mathbb{B}\to\mathbb{C}$ be a regular homeomorphic solution of the equation (3) which belongs to Sobolev class $W^{1,2}_{\rm loc}$, and normalized by $f(0)=0$ and $K>0$. Assume that the coefficient $\sigma:\mathbb{B}\to\mathbb{C}$ satisfies the following condition*

$$\frac{|\sigma(z)|^{m+2}}{(\operatorname{Im}\sigma(z))^{m+1}}\leqslant K\,|z|$$

for a.a. $z\in\mathbb{B}$. Then

$$\limsup_{z\to0}\frac{|f(z)|}{|z|}\geqslant K^{-\frac{1}{m}}.$$

Theorem 3. *Let $f:\mathbb{B}\to\mathbb{C}$ be a regular homeomorphic solution of the equation (3) which belongs to Sobolev class $W^{1,2}_{\rm loc}$, and normalized by $f(0)=0$. Suppose that*

$$\sigma_0=\liminf_{\varepsilon\to0}\frac{1}{\pi\varepsilon^2}\int\limits_{B_\varepsilon}\frac{|\sigma(z)|^{m+2}}{|z|\,(\operatorname{Im}\sigma(z))^{m+1}}\,dxdy.$$

1) If $\sigma_0\in(0,\infty)$, then

$$\limsup_{z\to0}\frac{|f(z)|}{|z|}\geqslant c_m\,\sigma_0^{-\frac{1}{m}},$$

where c_m is a positive constant depending on the parameter m.

2) If $\sigma_0=0$, then

$$\limsup_{z\to0}\frac{|f(z)|}{|z|}=\infty.$$

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On the type of Grassman image of a time-like minimal surface in Minkowski space

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In the Minkowski space ${}^1R_4$ there is a coordinate system in which the metric of the space has the form $ds^2 = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$. Let the equation $r = r(u^1, u^2)$ defines two-dimensional time-like surface F^2, the vectors ξ_1, ξ_2 are its space-like normal vectors, and g_{ij}, L_{ij}^k are the coefficients of the first and second quadratic forms, respectively. The number $H^k = g^{ij} L_{ij}^k$ is called the mean curvature of the surface for the direction of the normal vector ξ_k, and the vector $H = (H^1\xi_1 + H^2\xi_2)/2$ is the mean curvature vector. The time-like surfaces of Minkowski space with zero mean curvature vector will be called minimal surfaces, as in Euclidean space. We plan to apply the properties of the Grassman image of the minimal time-like surface to study its differential geometry, in particular, the question of the existence of such surfaces with some additional conditions on the Grassman image.

We can choose such a parameterization on the time-like surface F^2 in which $ds^2 = 2g_{12}du^1du^2$. It follows from the minimal surface condition that $L_{12}^k = 0$. Then the system of Gauss-Codazzi-Ricci equations takes the form

$$\begin{cases} R_{1212} = L_{11}^1 L_{22}^1 + L_{11}^2 L_{22}^2, \\ (L_{11}^1)'_{u^2} = -\mu_{12/2} L_{11}^2, \\ (L_{11}^2)'_{u^2} = \mu_{12/2} L_{11}^1, \\ (L_{22}^1)'_{u^1} = -\mu_{12/1} L_{22}^2, \\ (L_{22}^2)'_{u^1} = \mu_{12/1} L_{22}^1, \\ (\mu_{12/1})'_{u^2} - (\mu_{12/2})'_{u^1} + (L_{11}^1 L_{22}^2 - L_{11}^2 L_{22}^1)\frac{1}{g_{12}} = 0, \end{cases} \tag{1}$$

where $\mu_{12/i}$ are torsion coefficients. These equations coincide with the equations in the work [1].

The Grassman image of two-dimensional surfaces is their important geometric characteristic. The work [2] shows that the non-degenerated Grassman image Γ^2 of the surface of Minkowski space is two-dimensional surface $p = p(u^1, u^2)$, which belongs to the four-dimensional Grassman submanifold $PG(2,4)$ of six-dimensional pseudo-Euclidean space ${}^3R_6$ of index 3. Tangent vectors to Γ^2 can be written in the form $p_{u_i} = -L_{ik}^1 g^{kl}[r_l, \xi_2] - L_{ik}^2 g^{kl}[\xi_1, r_l], l = 1, 2$.

This paper shows that the metric of the Grassman image of the minimal time-like surface of the space ${}^1R_4$ with respect to the basis $e_1 = \frac{r_1 - r_2}{\sqrt{2g_{12}}}$, $e_2 = \frac{r_1 + r_2}{\sqrt{2g_{12}}}$, $e_3 = \xi_1$, $e_4 = \xi_2$ has the form $ds^2 = \frac{L_{11}^1 L_{22}^1 + L_{11}^2 L_{22}^2}{g_{12}} du^1 du^2$, and therefore the Grassman image of the minimal time-like surface is also the time-like surface.

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Open billiards, chaos and limit theorems

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Abstract: Chaos is one of the important subjects in the theory of dynamical systems. In 1958, Kolmogorov made a discovery regarding the statistical properties exhibited by certain chaotic dynamical systems.

I will talk about the relationship between chaotic billiard systems and their statistical properties. More precisely, I will show that

(1) Poisson limit theorems can characterize chaotic behaviors of billiard systems
(2) The convergence rates of Poisson limit theorems and Zeta-Functions have certain connections.

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On the inverse Poletsky inequality with a cotangent dilatation

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The following definitions are from [1]. A path γ in $\mathbb{R}^n$ is a continuous mapping $\gamma : \Delta \to \mathbb{R}^n$ where Δ is an interval in $\mathbb{R}$. Its locus $\gamma(\Delta)$ is denoted by $|\gamma|$. Given a family Γ of paths γ in $\mathbb{R}^n$, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for Γ, abbr. $\rho \in \operatorname{adm}\Gamma$, if

$$\int\limits_{\gamma} \rho(x)|dx| \geqslant 1$$

for each (locally rectifiable) $\gamma \in \Gamma$. Given $p \geqslant 1$, the *p-modulus* of Γ is defined by the relation

$$M_p(\Gamma) := \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^p(x)\,dm(x) \tag{1}$$

interpreted as $+\infty$ if $\operatorname{adm}\Gamma=\varnothing$.

We will need the following definitions related to paths, their lengths and mappings defined on them, see [2, section 8]. If $\gamma:\Delta\rightarrow{\mathbb R}^n$ is a locally rectifiable path, then there is the unique nondecreasing length function l_{γ} of Δ onto a length interval $\Delta_{\gamma}\subset{\mathbb R}$ with a prescribed normalization $l_{\gamma}(t_0)=0\in\Delta_{\gamma}$, $t_0\in\Delta$, such that $l_{\gamma}(t)$ is equal to the length of the subpath $\gamma|_{[t_0, t]}$ of γ if $t>t_0$, $t\in\Delta$, and $l_{\gamma}(t)$ is equal to minus length of $\gamma|_{[t, t_0]}$ if $t<t_0$, $t\in \Delta$. Let $g:|\gamma|\rightarrow {\mathbb R}^n$ be a continuous mapping, and suppose that the path $\widetilde{\gamma}=g\circ\gamma$ is also locally rectifiable. Then there is a unique non-decreasing function $L_{\gamma, g}:\Delta_{\gamma}\rightarrow \Delta_{\widetilde{\gamma}}$ such that $L_{\gamma, g}\left(l_{\gamma}\left(t\right)\right)=l_{\widetilde{\gamma}}(t)$ for all $t\in\Delta$. A path γ in D is called here a (whole) *lifting* of a path $\widetilde{\gamma}$ in ${\mathbb R}^n$ under $f:D\rightarrow{\mathbb R}^n$ if $\widetilde{\gamma}=f\circ\gamma$.

Further, we use the notation I for the segment $[a, b]$. Given a closed rectifiable path $\gamma:I\rightarrow{\mathbb R}^n$, we define a length function $l_{\gamma}(t)$ by the rule $l_{\gamma}(t)=S\left(\gamma, [a, t]\right)$, where $S(\gamma, [a, t])$ is the length of the path $\gamma|_{[a, t]}$. Let $\alpha:[a, b]\rightarrow{\mathbb R}^n$ be a rectifiable path in ${\mathbb R}^n$, $n\geqslant 2$, and $l(\alpha)$ be its length. A *normal representation* α^0 of α is defined as a path $\alpha^0:[0, l(\alpha)]\rightarrow {\mathbb R}^n$ which can be got from α by change of parameter such that $\alpha(t)=\alpha^0\left(S\left(\alpha, [a, t]\right)\right)$ for every $t\in[0, l(\alpha)]$. Such a normal representation always exists and is unique (see [1, Theorem 2.4]).

The following definition may be found in [1, 2.5, item 2, section I]. Let $\alpha:[a, b]\rightarrow{\mathbb R}^n$ be a closed rectifiable path in ${\mathbb R}^n$, $n\geqslant 2$. A mapping $f:|\alpha|\rightarrow{\mathbb R}^n$ is said to be *absolutely continuous on* α, if the function $f\circ\alpha^0$ is absolutely continuous on $[0, l(\alpha)]$, where $l(\alpha)$ denotes the length of α, and α^0 is its normal representation.

In the following, we say that some property P holds for *p-almost all paths in the domain D* if this property may be violated only for some family Γ_0 of paths in D such that $M_p(\Gamma_0)=0$, where $M_p(\Gamma_0)$ denotes the p-module of the family of paths Γ_0 defined in (1). We will say that the mapping $f:D\rightarrow{\mathbb R}^n$ has the *ACP-property with respect to p-modulus,* write $f\in ACP_p$, if the length function $L_{\gamma, f}$ is absolutely continuous on all closed intervals Δ_{γ} for p-almost all paths γ in D.

Let X and Y be two spaces with measures μ and $\mu^{\prime}$, respectively. We say that a mapping $f:X\rightarrow Y$ has *N-property of Luzin*, if from the condition $\mu(E)=0$ it follows that $\mu^{\prime}(f(E))=0$. Similarly, we say that a mapping $f:X\rightarrow Y$ has *N^{-1}-Luzin property,* if from the condition $\mu^{\prime}(E)=0$ it follows that $\mu(f^{-1}(E))=0$.

Let $x\in D$ be a differentiability point of f. We set

$$l\left(f^{\,\prime}(x)\right)\,=\,\min\limits_{h\in {\mathbb R}^n \backslash \{0\}} \frac {|f^{\,\prime}(x)h|}{|h|}\,,\quad \|f^{\,\prime}(x)\|\,=\,\max\limits_{h\in {\mathbb R}^n \backslash \{0\}} \frac {|f^{\,\prime}(x)h|}{|h|}\,,\quad J(x,f)=\det f^{\,\prime}(x)\,.$$

Given sets E and F and a given domain D in $\overline{{\mathbb R}^n}={\mathbb R}^n\cup\{\infty\}$, we denote by $\Gamma(E, F, D)$ the family of all paths $\gamma:[0, 1]\rightarrow \overline{{\mathbb R}^n}$ joining E and F in D, that is, $\gamma(0)\in E$, $\gamma(1)\in F$ and $\gamma(t)\in D$ for all $t\in(0, 1)$. Everywhere below, unless otherwise stated, the boundary and the closure of a set are understood in the sense of the extended Euclidean space $\overline{{\mathbb R}^n}$. Let $x_0\in\overline{D}$, $x_0\ne\infty$,

$$B(x_0, r)=\{x\in {\mathbb R}^n: |x-x_0|< r\}\,,\quad {\mathbb B}^n=B(0, 1)\,,$$

$$S(x_0,r) = \{x\,\in\,{\mathbb R}^n : |x-x_0|=r\}\,, S_i=S(x_0, r_i)\,,\quad i=1,2\,,$$

$$A=A(x_0, r_1, r_2)=\{x\,\in\,{\mathbb R}^n: r_1<|x-x_0|<r_2\}\,.$$

Let $f:D\rightarrow{\mathbb R}^n$, $n\geqslant 2$, and let $Q:{\mathbb R}^n\rightarrow [0, \infty]$ be a Lebesgue measurable function such that $Q(y)\equiv 0$ for $y\in{\mathbb R}^n\setminus f(D)$. Let $A=A(y_0, r_1, r_2)$ and $\Gamma_f(y_0, r_1, r_2)$ denotes the family of all paths $\gamma:[a, b]\rightarrow D$ such that $f(\gamma)\in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0,r_1,r_2))$, i.e., $f(\gamma(a))\in S(y_0, r_1)$,

$f(\gamma(b))\in S(y_0,r_2)$, and $f(\gamma(t))\in A(y_0,r_1,r_2)$ for any $a<t<b$. We say that f *satisfies the inverse Poletsky inequality at* $y_0\in f(D)$ *with respect to* p*-modulus,* if the relation

$$M_p(\Gamma_f(y_0,r_1,r_2))\leqslant\int\limits_A Q(y)\cdot\eta^p(|y-y_0|)\,dm(y) \tag{2}$$

holds for any $0<r_1<r_2<r_0:=\sup\limits_{y\in f(D)}|y-y_0|$ and any Lebesgue measurable function $\eta:(r_1,r_2)\to[0,\infty]$ such that $\int\limits_{r_1}^{r_2}\eta(r)\,dr\geqslant 1$. A mapping $f:D\to\mathbb{R}^n$ is called *weakly light,* if, for any $y\in\mathbb{R}^n$, each connected component $\{f^{-1}(y)\}$ does not contain a non-degenerate path (see, e.g., Remark 8.3 in [2]).

Theorem 1. *Let* $p>1$, *and let* $f:D\to\mathbb{R}^n$ *be a weakly light mapping which is differentiable a.e. and has Luzin* N*- and* N^{-1}*-properties with respect to the Lebesgue measure in* $\mathbb{R}^n$, *besides that,* $f\in ACP_p(D)$. *Let* $y_0\in\overline{f(D)}\setminus\{\infty\}$. *Set*

$$K_{CT,p,y_0}(y,f)=\sum\limits_{x\in f^{-1}(y)}\frac{\left(\sup\limits_{|h|=1}\left|\left(f'(x)h,\frac{f(x)-y_0}{|f(x)-y_0|}\right)\right|\right)^p}{|J(x,f)|}\,. \tag{3}$$

Then f *satisfies the inverse Poletsky inequality* 2 *at* y_0 *for* $Q_*(y):=K_{CT,p,y_0}(y,f)$.

The result mentioned above is published in [3].

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Hasse norm theorem for 3-manifolds

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Abstract:Following the analogies between knots and primes, 3-manifolds and number rings in arithmetic topology, we show a topological analogue of the Hasse norm principle for finite cyclic coverings of 3-manifolds, which was originally stated for finite cyclic extensions of number fields.

Theorem 1. *Let* M *be an integral homology* 3*-sphere endowed with a very admissible link* $\mathcal{L}$. *Let* $f:N\to M$ *be a finite cyclic covering branched over a finite sublink* L_0 *of* $\mathcal{L}$. *Then,*

$$P_{M,\mathcal{L}}\cap f_*(I_{N,f^{-1}(\mathcal{L})})=f_*(P_{N,f^{-1}(\mathcal{L})}).$$

Lemma 2. *Let M be an oriented connected closed 3-manifold endowed with a very admissible link $\mathcal{L}$. Let $f : N \to M$ be a finite covering branched over a finite link $L_0 \subset \mathcal{L}$. Let $f_* : I_{N,f^{-1}(\mathcal{L})} \to I_{M,\mathcal{L}}$ denote the homomorphism induced by f. Then, we have*

$$f_*\Big(\prod_{J\subset f^{-1}(\mathcal{L})} \mathbb{Z}[\mu_J]\Big) \subset \prod_{K\subset \mathcal{L}} \mathbb{Z}[\mu_K].$$

Proposition 3. *Let M be an integer homology 3-shpere endowed with a very admissible link $\mathcal{L}$ and $[A] \in H_2(M, \mathcal{L})$. Then there is a finite sublink $L \subset \mathcal{L}$ such that $[A] \in H_2(M, L)$. We can write $[A] = \sum_{K\subset L} c_K[S_K]$ with $c_K \in \mathbb{Z}$. Let $\Delta_{M,\mathcal{L}}([A]) = (a_K)_{K\subset\mathcal{L}} \in I_{M,\mathcal{L}}$. Then we have the following formula:*

$$a_K = \begin{cases} c_K[\lambda_K] - \Big(\sum_{K'\subset L\setminus K} \mathrm{lk}(K, K')c_{K'}\Big)[\mu_K] & (K \subset L) \\ -\sum_{K'\subset L} \mathrm{lk}(K, K')c_{K'}[\mu_K] & (K \subset \mathcal{L} \setminus L) \end{cases}$$

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A new Newton-type method and connections to Schroder theorem, Voronoi's diagrams, Newton's flows and the Riemann hypothesis

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The speaker has designed, very recently [4], a new Newton-type's method for root finding and optimization, which can be applied in any dimensions. The method is named Backtracking New Q-Newton's method (BNQN).

This talk concerns the application of this method to finding roots of a meromorphic function in 1 complex variable. I will present:

- The convergence guarantee theorem when applying BNQN to finding roots of meromorphic functions, from [5].

- The experiments from [4], which shows that usually the basins of attraction of BNQN are much more smooth than that of Newton's method. This is rather unexpected, given that BNQN depends on many seemingly random factors.

- The theorem from [2] which proves rigorously that the dynamics of BNQN, for finding roots of a polynomial of degree 2, is the same as the classical Schrőder's theorem for dynamics of Newton's method (except that BNQN is not chaotic on the boundary line).

- Experiments from [1] which reveal some surprising connections between BNQN and Voronoi's diagrams and Newton's flows.

- New results from [3] which connects the dynamics of BNQN and the Riemann hypothesis.

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Geometric and algebraic properties of dispersionless Nizhnik equation

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The dispersionless Nizhnik equation (see [1] for justifying this name)

$$u_{txy} = (u_{xx}u_{xy})_x + (u_{xy}u_{yy})_y \tag{1}$$

is the dispersionless limit of the symmetric Nizhnik equation, which is the potential equation of the Nizhnik system [3] in the symmetric case. The equation (1) has interesting geometric and algebraic properties. In particular, the maximal Lie invariance (pseudo)algebra $\mathfrak{g}$ of (1) is infinite-dimensional and is spanned by the vector fields

$$\begin{aligned}
&D^t(\tau) = \tau\partial_t + \tfrac{1}{3}\tau_t x\partial_x + \tfrac{1}{3}\tau_t y\partial_y - \tfrac{1}{18}\tau_{tt}(x^3+y^3)\partial_u, \quad D^{\rm s} = x\partial_x + y\partial_y + 3u\partial_u,\\
&P^x(\chi) = \chi\partial_x - \tfrac{1}{2}\chi_t x^2\partial_u, \quad P^y(\rho) = \rho\partial_y - \tfrac{1}{2}\rho_t y^2\partial_u,\\
&R^x(\alpha) = \alpha x\partial_u, \quad R^y(\beta) = \beta y\partial_u, \quad Z(\sigma) = \sigma\partial_u,
\end{aligned}$$

where τ, χ, ρ, α, β and σ run through the set of smooth functions of t. Moreover, the contact invariance (pseudo)algebra $\mathfrak{g}_{\rm c}$ of (1) coincides with the first prolongation of the algebra $\mathfrak{g}$.

The point- and contact-symmetry pseudogroups G and $G_{\rm c}$ of (1) were efficiently constructed in [1] by using the original version of the algebraic megaideal-based method suggested in [2]. The basic (necessary) method condition that the pushforward Φ_* of elements $\mathfrak{g}$ by any element Φ of G preserves any megaideal $\mathfrak{m}$ of $\mathfrak{g}$, $\Phi_*\mathfrak{m} \subseteq \mathfrak{m}$, is replaced in this version by a weaker but more computationally efficient condition $\Phi_*(\mathfrak{m}\cap\mathfrak{s})\subseteq\mathfrak{m}$ for an arbitrary essential megaideal $\mathfrak{m}$ and a selected fixed finite-dimensional subalgebra $\mathfrak{s}$ of $\mathfrak{g}$. As such $\mathfrak{s}$ for (1), we can take $\mathfrak{s}_1=\mathfrak{a}\ni\langle D^{\rm s}\rangle$ or $\mathfrak{s}_2=\mathfrak{a}\ni\langle D^t(1),D^t(t)\rangle$, where $\mathfrak{a}=\big\langle Z(1),Z(t),R^z(1),P^z(1),P^z(t),z\in\{x,y\}\big\rangle$.

Theorem 1. *(i) The point-symmetry pseudogroup G of the dispersionless Nizhnik equation (1) is generated by the transformations of the form*

$$\tilde t=T(t),\quad \tilde x=CT_t^{1/3}x+X^0(t),\quad \tilde y=CT_t^{1/3}y+Y^0(t),$$
$$\tilde u=C^3u-\frac{C^3T_{tt}}{18T_t}(x^3+y^3)-\frac{C^2}{2T_t^{1/3}}(X^0_tx^2+Y^0_ty^2)+W^1(t)x+W^2(t)y+W^0(t)$$

and the transformation $\mathscr{J}$: $\tilde t=t$, $\tilde x=y$, $\tilde y=x$, $\tilde u=u$. Here T, X^0, Y^0, W^0, W^1 and W^2 are arbitrary smooth functions of t with $T_t\neq0$, and C is an arbitrary nonzero constant.

(ii) The contact-symmetry pseudogroup $G_{\rm c}$ of the dispersionless Nizhnik equation (1) coincides with the first prolongation $G_{(1)}$ of the pseudogroup G.

Thus, a complete list of independent discrete point symmetry transformations of (1) is exhausted by three commuting involutions, $\mathscr{J}$, $\mathscr{I}$ and $\mathscr{I}$, which map (t,x,y,u) to (t,y,x,u), $(-t,-x,-y,u)$ and $(t,-x,-y,-u)$, respectively.

The equation (1) is peculiar due to the fact that the condition $\Phi_*\mathfrak{g}\subseteq\mathfrak{g}$ completely defines G and thus is not only necessary but also sufficient in this particular case. The similar claim holds for $\mathfrak{g}_{\rm c}$ and $G_{\rm c}$. This is the first and so far the only example of this kind in the literature.

In the context of the method applied, an important problem is to select certain subalgebras of $\mathfrak{g}$ and $\mathfrak{g}_{\rm c}$.

Definition 2. We call a proper subalgebra $\mathfrak{s}$ of a Lie algebra $\mathfrak{a}$ of vector fields a *subalgebra defining the diffeomorphisms that stabilize* $\mathfrak{a}$ if the conditions $\Phi_*\mathfrak{a}\subseteq\mathfrak{a}$ and $\Phi_*\mathfrak{s}\subseteq\mathfrak{a}$ for local diffeomorphisms Φ in the underlying space are equivalent.

Theorem 3. *The subalgebra $\mathfrak{s}_2$ of the algebra $\mathfrak{g}$ defines the diffeomorphisms that stabilize $\mathfrak{g}$, whereas the subalgebra $\mathfrak{s}_1$ and even the subalgebra $\bar{\mathfrak{s}}_1:=\mathfrak{s}_1+\langle D^t(1)\rangle$ does not have this property.*

Corollary 4. *The first prolongation of $\mathfrak{s}_2+\mathfrak{s}_3$ with $\mathfrak{s}_3:=\langle Z(1),Z(t),Z(t^2),R^z(1),R^z(t),z\in\{x,y\}\rangle$, which is a subalgebra of $\mathfrak{g}_{\rm c}=\mathfrak{g}_{(1)}$, defines the diffeomorphisms of the corresponding first-order jet space that stabilize $\mathfrak{g}_{\rm c}$.*

We also found geometric properties of the dispersionless Nizhnik equation (1) that completely define this equation. Although the maximal Lie invariance algebra $\mathfrak{g}$ of the equation (1) exhaustively defines the point-symmetry pseudogroup G of this equation, it does not define exhaustively the equation itself.

Lemma 5. *(i) A partial differential equation of order less than or equal to three with three independent variables is invariant with respect to the algebra $\mathfrak{g}$ if and only if it is of the form*

$$u_{txy}=(u_{xx}u_{xy})_x+(u_{xy}u_{yy})_y+u_{xy}u_{xyy}H(\omega_1,\omega_2),\quad \omega_1:=\frac{u_{xxx}-u_{yyy}}{u_{xyy}},\quad \omega_2:=\frac{u_{xxy}}{u_{xyy}},\tag{2}$$

where H is an arbitrary smooth function of its arguments.

(ii) An equation of the form (2) admits the conservation-law characteristic 1 and thus it is in conserved form if and only if H is an affine function of (ω_1,ω_2), i.e., $H=a\omega_1+b\omega_2+c$ for some constants a, b and c, and the equation takes the form

$$u_{txy}=(u_{xx}u_{xy})_x+(u_{xy}u_{yy})_y+u_{xy}\big(a(u_{xxx}-u_{yyy})+bu_{xxy}+cu_{xyy}\big).\tag{3}$$

(iii) An equation of the form (3) admits the conservation-law characteristic u_{xx} *or* u_{yy} *if and only if* $a = b = 0$ *or* $a = c = 0$*, respectively.*

Theorem 6. *An* r*th order* ($r \in \{1, 2, 3\}$) *partial differential equation with three independent variables admits the algebra* $\mathfrak{g}$ *as its Lie invariance algebra and the conservation-law characteristics* 1, u_{xx} *and* u_{yy} *if and only if it coincides with the dispersionless Nizhnik equation* (1).

The presented properties of the equation (1) are used in [4] to construct its exact solutions.

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Chain-regular and regular components of the wandering set of surface homeomprphisms

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Regular components of the wandering set of surface homeomorphisms were introduced by Birkgoff [1, 2]. With the emergence of the chain recurrent set theory introduced by Conley [3] for flows and adapted for discrete dynamical systems by Franks and Hurley [4, 5] we can define an analog of regular components of the wandering set for the set of chain-regular points (points that are not chain recurrent) as the set of points that divide an attractor-repeller pair.

We study the topology of chain-regular components of surface homeomorphisms and show that it is in fact different from the topology of regular components of the wandering set.

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Dynamics of influenza with the rates of vaccination and treatment

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Influenza is one of the most common diseases worldwide. In this work, we investigate the dynamics of influenza effected by vaccination and treatment with an SIR model that includes Caputo type fractional derivative. These dynamics are explained with this model by using Fractional Backward Euler Method [1].

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Topology of the Hilbert Schemes of monomial plane curve singularities

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Let X be a singular irreducible plane curve over $\mathbb{C}$. For a singular point o of X, we refer to the pair (X, o) as a plane curve singularity. Let $\mathcal{O}_{X,o}$ (resp. Γ) be the local ting of (X, o) (resp. the semi-group associated with (X, o)). We denote by $\mathrm{Hilb}^r(X, o)$ the punctual Hilbert scheme of degree r for a given singularity (X, o). Piontkowski [1] studied the topology of the Jacobian factor $J_{X,o}$ for a plane curve singularity (X, o) with $\Gamma = \langle p, q\rangle$ ($\gcd(p, q) = 1$). He showed the existence of an affine cell decomposition of the Jocobi factor $J_{X,o}$ and the Euler number of J_X and the Betti numbers of J_X are described. In this talk, we generalize Piontkowski's results to the cases of the punctual Hilbert schemes of (X, o).

In this talk, we always consider the plane curve singularity whose local ring $\mathcal{O}_{X,o}$ is $\mathbb{C}[[t^p, t^q]]$ where $\gcd(p, q) = 1$. Then such a singularity has $\Gamma = \langle p, q\rangle$ as its semi-group. Let $\mathrm{Mod}(\Gamma)$ be the set of all Γ-semi-modules. Defining $\mathrm{codim}\Delta := \#(S \setminus \Delta)$, we set $\mathrm{Mod}^r(\Gamma) := \{\Delta \in \mathrm{Mod}(\Gamma) | \,\mathrm{codim}\Delta = r\}$.

It is known that the components of $\mathrm{Hilb}^r(X,o)$ is parametrized by the elements of $\mathrm{Mod}^r(\Gamma)$.

$$\mathrm{Hilb}^r(X,o) = \bigcup_{\Delta \in \mathrm{Mod}^r(\Gamma)} H(\Delta) \tag{1}$$

The component $H(\Delta)$ in (1) is called the Δ-subset of $\mathrm{Hilb}^r(X,o)$.

Theorem 1. *Let (X,o) be a plane curve singularity whose local ring $\mathcal{O}_{X,o}$ is $\mathbb{C}[[t^p,t^q]]$ where $\gcd(p,q)=1$. Each Δ-subset $H(\Delta)$ in (1) is isomorphic to an affine space whose dimension is given by*

$$\sum_{i=1}^{p-1} \#[\{(\Gamma - \min \Delta) \cap [a_i, a_i + q]\} \setminus \Delta^{(0)}]. \tag{2}$$

Here $\Delta^{(0)}$ is the 0-nomalization of Δ and $\{a_0,\dots,a_{p-1}\}$ is the p-basis of $\Delta^{(0)}$.

The following fact follows from Theorem 1

Corollary 2. *Let (X,o) be a plane curve singularity with $\mathcal{O}_{X,o} = \mathbb{C}[[t^p,t^q]]$ $(\gcd(p,q)=1)$. The Euler number of $\mathrm{Hilb}^r(X,o)$ is equal to $\#\mathrm{Mod}^r(\Gamma)$.*

We denote by $e(\mathrm{Hilb}^r(X,o))$ the Euler number of $\mathrm{Hilb}^r(X,o)$.

Example 3. The Euler numbers of the punctual Hilbert schemes for the A_{2l}-singularity are given in the following table:

r	$0 \le r \le 2l$	$r \ge 2l+1$
$e(\mathrm{Hilb}^r(X,o))$	$[r/2]+1$	$l+1$

Here the notaion $[a]$ $(a \in \mathbb{R})$ is the biggest integer that is smaller than a.

Setting $\operatorname{codim} H(\Delta) := \dim \mathrm{Hilb}^r(X,o) - \dim\, H(\Delta)$, we define

$$\mathcal{H}_{r,d} := \{H(\Delta) \,|\, \Delta \in \mathrm{Mod}^r(\Gamma) \text{ and } \dim H(\Delta) = d\},$$
$$\mathcal{H}_r^d := \{H(\Delta) \,|\, \Delta \in \mathrm{Mod}^r(\Gamma) \text{ and } \operatorname{codim} H(\Delta) = d\}.$$

Theorem 4. *Let (X,o) be a plane curve singularity with the local ring $\mathbb{C}[[t^p,t^q]]$ $(\gcd(p,q)=1)$. Then the odd (co-) homology groups of $\mathrm{Hilb}^r(X,o)$ are zero. The even (co-) homology groups of $\mathrm{Hilb}^r(X,o)$ are free abelian groups with Betti numbers*

$$h_{2d}(\mathrm{Hilb}^r(X,o)) = \#\mathcal{H}_{r,d} \text{ and } h^{2d}(\mathrm{Hilb}^r(X,o)) = \#\mathcal{H}_r^d.$$

Example 5. The even (co-) homology groups of $\mathrm{Hilb}^r(X,o)$ for the A_{2l}-singularity are given in the following table:

r	$0 \le r \le 2l$	$r \ge 2l+1$
d	$0 \le d \le r$	$0 \le d \le l$
$h_{2d}(\mathrm{Hilb}^r(X,o))$	1	1
$h^{2d}(\mathrm{Hilb}^r(X,o))$	1	1

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Self-similar actions of the fundamental group of the Klein bottle

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A virtual endomorphism of a group G is a homomorphism of the form $\phi : H \to G$, where $H < G$ is a subgroup of finite index. A virtual endomorphism $\phi : H \to G$ is called simple if there are no nontrivial normal ϕ-invariant subgroups.

A recursive construction using a simple virtual endomorphism ϕ produces a so-called self-similar action of the group G on a d-regular rooted tree X^*. The X^* represents words over the alphabet X of size d. In general, a faithful action of a group G on rooted tree X^* is said to be *self-similar* if for every $g \in G$ and every $x \in X$ there exists unique pair $g|_x \in G$ and $y \in X$ such that $g(xw) = yg|_x(w)$. A self-similar action is called *self-replicating* if the associative simple virtual endomorphism ϕ is surjective. One can find more information regarding self-similar actions in [1].

Consider the fundamental group of the Klein bottle K. The group K is finitely generated by affine transformations $t(x, y) = (x, y + 1)$ and $s(x, y) = (x + 1/2, -y)$. We can show that for every virtual endomorphism $\phi : H \to K$ there exist subgroup of finite index $H_1 \sim \mathbb{Z}^2$ and associated matrix $B_\phi \in M_2(\mathbb{Q})$ of rational entities such that the restriction $\phi|_{H_1} : H_1 \to K$ is in fact a linear map $\phi|_{H_1}(x) = B_\phi x$.

Theorem 1. *Let $\phi : H \to K$ be a virtual endomorphism and $B_\phi \in M_2(\mathbb{Q})$ the associated matrix. Then ϕ is simple, and therefore produces a self-similar action, if and only if B_ϕ is not of the forms:*

$$\begin{pmatrix} \alpha & \frac{n}{m}\beta \\ \frac{m}{n}\gamma & \delta \end{pmatrix}, \frac{1}{2}\begin{pmatrix} \alpha+\beta+\gamma+\delta & \frac{n}{m}(\alpha+\beta-\gamma-\delta) \\ \frac{m}{n}(\alpha-\beta+\gamma-\delta) & \alpha-\beta-\gamma+\delta \end{pmatrix}, \begin{pmatrix} k & b_1 \\ 0 & b_2 \end{pmatrix} \text{ or } \begin{pmatrix} b_1 & 0 \\ b_2 & k \end{pmatrix} \tag{1}$$

for $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$; $n, m, k \in \mathbb{Z}$; $b_i \in \mathbb{Q}$.

Theorem 2.
1) *The group K admits a transitive self-similar action on a d-regular rooted tree if and only if $d \geq 2$ is not an odd prime.*
2) *The group K admits a self-replicating action on a d-regular rooted tree if and only if d is not a prime or a power of 2.*

A self-similar action (G, X^*) is called *finite-state* if for every $g \in G$ the set of its sections $\{g|_v : v \in X^*\}$ is finite. In other words, the action of every element can be emulated by a finite-state transducer.

A self-similar action (G, X^*) is called *contracting* if there exists finite set $\mathcal{N} \subset G$ such that for every $g \in G$ there exists $n \in \mathbb{N}$ such that $g|_v \in \mathcal{N}$ for all $v \in X^*$ of length $\geq n$.

Theorem 3. *Let (K, X^*) be a transitive self-similar action and B_ϕ the matrix of the associated virtual endomorphism ϕ.*

1) *The (K, X^*) is contracting if and only if the eigenvalues of B_ϕ are less than 1 in modulus.*
2) *If (K, X^*) is self-replicating, then (K, X^*) is finite-state if and only if it is contracting.*
3) *The group K admits a transitive finite-state (contracting) action of degree d if and only if $d \geq 2$ is not an odd prime.*

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Applications of dimension theory to embeddability problems in topological data analysis: the case study of the Gromov-Hausdorff distance

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Topological data analysis (TDA) is a recent and fast-growing subject aiming to apply topological techniques to a wide range of applications. Areas where topological data analysis has found applications include neuroscience, image recognition, biology and evolutionary networks. Treating datasets or topologically inspired invariants assigned to them as points of a metric space is a central idea. However, comparing objects in this new metric space is often computationally challenging. Therefore, a technique usually deployed is mapping those objects in a Hilbert space using *vectorisation methods* or *kernels* to allow their implementation in machine learning pipelines. The quest to construct such maps that distort the distances in a controlled way is a crucial research area.

In this talk, we present how dimension theory can fruitfully assist this search. Regarding these maps as examples of certain metric embedding classes is the key-viewpoint shift. As a leading example, we discuss embeddability results for families of metric spaces endowed with the Gromov-Hausdorff distance. In addition to the intrinsic interest of these results given by the importance of this metric in Riemannian geometry and geometric group theory, they also impact TDA since the Gromov-Hausdorff distance was recently proposed as a theoretical framework for shape and dataset comparisons. More precisely, we show the following results:

- the space $\mathcal{GH}_n$ of metric spaces with at most n elements can be coarsely embedded into a Hilbert space;
- their union $\bigcup_{n\in\mathbb{N}}\mathcal{GH}_n$ cannot be coarsely embedded into any Hilbert space;
- the space of metric spaces whose diameter is bounded by r cannot be bi-Lipschitz embedded into any $\mathbb{R}^m$.

If time permits, we conclude by presenting results concerning the embeddability of persistence diagrams, one of the cornerstone notions of TDA, and periodic point sets, a generalisation of lattices used to represent crystals in material science and pharmaceutics.

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Balayage on locally compact spaces

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This talk is based on [1]–[3], and it is devoted to the theory of potentials on a locally compact (Hausdorff) space X with respect to a *kernel* κ, κ being thought of as a symmetric, lower semicontinuous function $\kappa : X\times X \to [0,\infty]$. To be exact, we are interested in generalizations of the classical theory of balayage (sweeping out) on $\mathbb{R}^n$, $n\geqslant 2$ (see e.g. [4]–[7]), to a suitable kernel κ on X.

We denote by $\mathfrak{M}$ the linear space of all (real-valued Radon) measures μ on X, equipped with the *vague* topology of pointwise convergence on the continuous functions $\varphi : X \to \mathbb{R}$ of compact support, and by $\mathfrak{M}^+$ the cone of all positive $\mu \in \mathfrak{M}$. (For the theory of measures and integration on X, we refer to Bourbaki [8].) Given $\mu, \nu \in \mathfrak{M}$, the *mutual energy* and the *potential* are introduced by

$$I(\mu,\nu) := \int \kappa(x,y)\, d(\mu \otimes \nu)(x,y) \quad \text{and} \quad U^\mu(x) := \int \kappa(x,y)\, d\mu(y), \quad x \in X,$$

respectively, provided the value on the right is well defined as a finite number or $\pm\infty$. For $\mu = \nu$, the mutual energy $I(\mu,\nu)$ defines the *energy* $I(\mu,\mu) =: I(\mu)$ of $\mu \in \mathfrak{M}$.

In what follows, a kernel κ is assumed to satisfy the *energy principle*, which means that $I(\mu) \geqslant 0$ for all (signed) $\mu \in \mathfrak{M}$, and moreover that $I(\mu) = 0$ only for $\mu = 0$. Then all $\mu \in \mathfrak{M}$ of finite energy form a pre-Hilbert space $\mathcal{E}$ with the inner product $\langle \mu, \nu \rangle := I(\mu,\nu)$ and the energy norm $\|\mu\| := \sqrt{I(\mu)}$, cf. [9, Lemma 3.1.2]. The topology on $\mathcal{E}$ introduced by means of this norm is said to be *strong*.

In addition, we shall always assume that κ satisfies the *consistency* principle, which means that the cone $\mathcal{E}^+ := \mathcal{E} \cap \mathfrak{M}^+$ is *complete* in the induced strong topology, and that the strong topology on $\mathcal{E}^+$ is *finer* than the vague topology on $\mathcal{E}^+$; such a kernel is said to be *perfect* (Fuglede [9]). Thus any strong Cauchy net $(\mu_j) \subset \mathcal{E}^+$ converges *both strongly and vaguely* to the same unique measure $\mu_0 \in \mathcal{E}^+$.

Yet another permanent requirement on κ is that it satisfies the *domination principle*, which means that for any $\mu \in \mathcal{E}^+$ and any $\nu \in \mathfrak{M}^+$ with $U^\mu \leqslant U^\nu$ μ-a.e., the same inequality holds on all of X.

For any $A \subset X$, we denote by $\mathfrak{C}_A$ the upward directed set of all compact subsets K of A, where $K_1 \leqslant K_2$ if and only if $K_1 \subset K_2$. If a net $(x_K)_{K \in \mathfrak{C}_A} \subset Y$ converges to $x_0 \in Y$, Y being a topological space, then we shall indicate this fact by writing: $x_K \to x_0$ in Y as $K \uparrow A$.

Given $A \subset X$, we denote by $\mathfrak{M}^+_A$ the class of all $\mu \in \mathfrak{M}^+$ *concentrated on* A, which means that $X \setminus A$ is locally μ-negligible, or equivalently that A is μ-measurable and $\mu = \mu|_A$, $\mu|_A$ being the trace of μ to A. Also write $\mathcal{E}^+_A := \mathfrak{M}^+_A \cap \mathcal{E}$, and let $\mathcal{E}'_A$ stand for the closure of $\mathcal{E}^+_A$ in the strong topology on $\mathcal{E}$. Being a strongly closed subcone of the strongly complete cone $\mathcal{E}^+$, $\mathcal{E}'_A$ is likewise strongly complete.

Denoting by $c_*(E)$ and $c^*(E)$ the *inner* and *outer* capacity of $E \subset X$, respectively [9, Section 2.3], we say that an assertion $\mathcal{A}(x)$ involving a variable point $x \in X$, holds *nearly everywhere* (*n.e.*), resp. *quasi-everywhere* (*q.e.*), on a set A if $c_*(E) = 0$, resp. $c^*(E) = 0$, where $E := \{x \in A : \ \mathcal{A}(x) \text{ fails}\}$.

Theorem 1. *For any $A \subset X$ and any $\sigma \in \mathcal{E}^+$, there exists $\sigma^A \in \mathcal{E}'_A$, called the inner balayage of σ to A, that is uniquely characterized by any one of the following (equivalent) assertions.*

(i) *σ^A is the unique measure in $\mathcal{E}'_A$ having the property*

$$U^{\sigma^A} = U^\sigma \quad \textit{n.e. on } A.$$

(ii) *σ^A is the unique orthogonal projection of σ in the pre-Hilbert space $\mathcal{E}$ onto the (convex, strongly complete) cone $\mathcal{E}'_A$, that is, $\sigma^A \in \mathcal{E}'_A$ and*

$$\|\sigma - \sigma^A\| = \min_{\mu \in \mathcal{E}'_A} \|\sigma - \mu\|.$$

(iii) *σ^A is the unique measure in $\mathcal{E}^+$ satisfying any one of the following three limit relations:*

$$\sigma^K \to \sigma^A \quad \textit{strongly in } \mathcal{E}^+ \textit{ as } K \uparrow A,$$

$$\sigma^K \to \sigma^A \quad \textit{vaguely in } \mathcal{E}^+ \textit{ as } K \uparrow A,$$

$$U^{\sigma^K} \uparrow U^{\sigma^A} \quad \textit{pointwise on } X \textit{ as } K \uparrow A,$$

where σ^K *denotes the only measure in* $\mathcal{E}^+_K$ *with the property* $U^{\sigma^K} = U^\sigma$ *n.e. on* K.

(iv) σ^A *is the only measure in the class* $\Gamma_{A,\sigma}$ *having the property*

$$U^{\sigma^A} = \min_{\mu \in \Gamma_{A,\sigma}} U^\mu \quad \textit{on all of } X, \tag{1}$$

where $\Gamma_{A,\sigma} := \bigl\{\mu \in \mathcal{E}^+ : \; U^\mu \geqslant U^\sigma \quad \text{n.e. on } A\bigr\}$.

(v) σ^A *is the only measure in the class* $\Gamma_{A,\sigma}$, *introduced above, with the property*

$$\|\sigma^A\| = \min_{\mu \in \Gamma_{A,\sigma}} \|\mu\|.$$

Theorem 2. *If a space* X *is second-countable, while a set* A *is Borel, then Theorem* 1 *remains valid with "n.e. on* A*" replaced throughout by "q.e. on* A*". The measure* ω^{*A}*, thereby uniquely determined, is said to be the outer balayage of* ω *onto* A*. Actually,* $\omega^{*A} = \omega^A$*.* (Compare with [10, Theorem 4.12].)

Remark 3. All the above-mentioned assumptions on a space X and a kernel κ are fulfilled by:

- ✓ The α-Riesz kernels $|x-y|^{\alpha-n}$ of order $\alpha \in (0, 2]$, $\alpha < n$, on $\mathbb{R}^n$, $n \geqslant 2$ (see [6, Theorems 1.15, 1.18, 1.27, 1.29]).
- ✓ The associated α-Green kernels, where $\alpha \in (0, 2]$ and $\alpha < n$, on an arbitrary open subset of $\mathbb{R}^n$, $n \geqslant 2$ (see [11, Theorems 4.6, 4.9, 4.11]).
- ✓ The (2-)Green kernel on a planar Greenian set (see [5] and [7, Sections I.V.10, I.XIII.7]).

(We emphasize that some of the results formulated above are new even for these classical kernels.)

Remark 4. The theory of balayage thereby developed has already been shown to be a powerful tool in the well-known Gauss variational problem, see [12].

Problem 5. What kind of additional assumptions on X and κ would make it possible to generalize the theory of balayage, presented above, to Radon measures on X of *infinite* energy?

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Узагальнені аналоги теореми Яно-Вестлейка

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Одним з важливих напрямків сучасної диференціальної геометрії є теорія афінорних структур на диференційовних многовидах, а також дифеоморфізми таких многовидів. В 1980 році американський геометр А.Грей отримав класифікацію майже комплексних структур на ріманових просторах [1]. Вона містить 16 класів, серед яких відомі келерова, K-, H-структури та інші, які привертали увагу багатьох сучасних математиків.

В теорії геодезичних відображень [2] відома теорема Яно-Вестлейка,яка стверджує, що келерові простори не допускають нетривіальних геодезичних відображень, що зберігають структуру.

Розглянемо геодезичне відображення ріманових просторів

$$f:(V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h),$$

на яких окрім метричних тензорів g_{ij}, $\overline{g}_{ij}$ задано афінори F_i^h, $\overline{F}_i^h$.

Основні рівняння геодезичного відображення зі збереженням структури в загальній за відображенням системі координат (x) мають вигляд:

$$\overline{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x) + \psi_i(x)\delta_j^h(x) + \psi_j(x)\delta_i^h(x),$$

$$F_i^h(x) = \overline{F}_i^h(x), \qquad h, i, j - 1, 2, ..., n,$$

де Γ_{ij}, $\overline{\Gamma}_{ij}^h$ - компоненти об'єктів зв'язності просторів V_n,$\overline{V}_n$ відповідно; ψ_i - деякий ковектор.

Афінорна структура на V_n називається майже комплексною ермітовою, якщо

$$F_\alpha^h F_i^\alpha = -\delta_i^h,$$

$$F_{ij} + F_{ji} = 0, \qquad F_{ij} = g_{i\alpha} F_j^\alpha.$$

Якщо при цьому коваріантна похідна афінора зодовольняє одній з умов

$$F_{i,j}^h = 0,$$

$$F_{i,j}^h + F_{j,i}^h = 0,$$

$$F_{i,j}^h + F_{\alpha,\beta}^h F_i^\alpha F_j^\beta = 0,$$

або

$$F_{i,j}^h + F_{\alpha,\beta}^h F_i^\alpha F_j^\beta + F_{j,i}^h + F_{\alpha,\beta}^h F_j^\alpha F_i^\beta = 0,$$

то простір називається келеровим, K-, O^*- або G_1-простором, відповідно (за класифікацією А.Грея).

Нами доведена

Теорема 1. *K-, O^*- і G_1-простори не допускають нетривіальних геодезичних відображень зі збереженням структури.*

Очевидно, ця теорема є узагальненням теореми Яно-Вестлейка.

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Геодезичні відображення псевдоріманових просторів

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Необхідною і достатньою умовою того, щоб псевдоріманів простір V_n допускав нетривіальні геодезичні відображення є існування в ньому розв'язків систем диференціальних рівнянь в коваріантних похідних

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} \tag{1}$$

відносно тензора $a_{ij} (= a_{ji} \neq c g_{ij})$ та вектора λ_i $(\neq 0)$.

Тут кома знак коваріантної похідної

$$a_{ij,k} = \partial_k a_{ij} - a_{\alpha j} \Gamma^{\alpha}_{ik} - a_{\alpha i} \Gamma^{\alpha}_{jk}.$$

Систему (1) називають *лінійною формою основних рівнянь теорії геодезичних відображень*. При відомих розв'язках системи (1) метричні тензори псевдоріманових просторів V_n та $\bar{V}_n$ можуть бути знайдені з рівнянь [1, 2]

$$a_{ij} = e^{2\varphi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j};$$
$$\lambda_i = -e^{2\varphi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta}.$$

Тут $\bar{g}_{ij}$ — елементи оберненої матриці до метричного тензору V_n.

Об'єкти псевдоріманового простору V_n, які визначені за допомогою метричного тензора g_{ij}, називають *внутрішніми об'єктами псевдоріманового простору.* Крім внутрішніх об'єктів вивчають і об'єкти, які не є внутрішніми, зокрема тензор D^h_{ijk} такий, що

$$D^h_{ijk} = R^h_{ijk} - B(\delta^h_k g_{ij} - \delta^h_j g_{ik}),$$

де δ^h_i — символи Кронекера, R_{ijk} —тензор Рімана, а B — деякий інваріант.

Якщо тензор $D^h_{ijk} = 0$, то псевдоріманів простір V_n є простором сталої кривини і

$$B = \frac{R}{n(n-1)}. \tag{2}$$

Тут R — скалярна кривина, яка визначається за формулою

$$R = R^{\alpha}_{\beta\gamma\alpha} g^{\beta\gamma}.$$

І навпаки, якщо виконується умова (2), то тензор D^h_{ijk} співпадає з тензором конциркулярної кривини, який визначається за формулою

$$Y^h_{ijk} = R^h_{ijk} - \frac{R}{n(n-1)}\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right).$$

Псевдоріманові простори, в яких існує тензор $T^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_m}$ такий, що

$$T^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_n, k} = \rho_k T^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_m} \tag{3}$$

називають *T-рекурентними.*

А якщо умови (3) виконуються для тензора Рімана, то такі простори називають *рекурентними.*

Векторні поля $u_k \neq 0$, які задовольняють для ненульових тензорів $T^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_m}$ умові

$$u_k T^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_m} + u_{j_1} T^{i_1 i_2 \dots i_r}_{j_2 k \dots j_m} + u_{j_2} T^{i_1 i_2 \dots i_r}_{k j_1 \dots j_n} = 0 \tag{4}$$

називають *векторними оболонками тензора* $T^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_m}$ відносно індексів j_1 та j_2.

Якщо векторна оболонка задається відносно кососиметричної пари індексів тензора Рімана, тобто

$$u_i R^h_{jkl} + u_k R^h_{jli} + u_l R^h_{jik} = 0, \tag{5}$$

то вона називається *векторною оболонкою тензора Рімана.*

Враховуючи властивість тензора Рімана

$$R^h_{ijk,l} + R^h_{ikl,j} + R^h_{ilj,k} = 0,$$

легко переконатись, що в рекурентних псевдоріманових просторах існує векторна оболонка тензора Рімана. Тому псевдоріманові простори, в яких виконуються умови (5), називають *слабо рекурентними просторами.* Якщо умові (4) буде задовольнять тензор D^h_{ijk}, тобто

$$u_i D^h_{jkl} + u_k D^h_{jli} + u_l D^h_{jik} = 0,$$

то такі простори будемо називат *D-слабо рекурентними псевдорімановими просторами.*

Якщо D-слаборекурентний псевдоріманів простір V_n допускає нетривіальні геодезичні відображення, то він або простір Ейнштейна, або

$$a_{\alpha i} u^\alpha = \tau u_i.$$

Просторами Ейнштейна називають псевдоріманові простори, в яких виконуються умови

$$R_{ij} = \frac{R}{n} g_{ij}.$$

Простори Ейнштейна, які характеризуються умовами на тензор Річчі мають велике значення, як в рімановій геометрії, так і в її застосуваннях [3, 4].

Чотиривимірні простори Ейнштейна V_4, відмінні від просторів сталої кривини, не допускають нетривіальні геодезичні відображення на псевдоріманові простори $\bar{V}_4$.

Таким чином, в чотирьохмірних D-слаборекурентних псевдоріманових просторах, відмінних від просторів сталої кривини, які допускають нетривіальні геодезичні відображення, вектор, що задає векторну оболонку, є власним вектором тензора a_{ij} із лінійної форми основних рівнянь теорії геодезичних відображень.

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Спеціальні келерові простори

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Келеровим простором K_n $(n = 2N)$ називається псевдоріманів простір з метричним тензором $g_{ij}(x)$, у якому існує структура $F_i^h(x)$, що задовольняє співвідношенням [1]:

$$F_\alpha^h F_i^\alpha = -\delta_i^h; \qquad F_{(ij)} = 0; \qquad F_{i,j}^h = 0,$$

де $F_{ij} \equiv g_{i\alpha} F_j^\alpha$, кома — знак коваріантної похідної по зв'язності K_n.

Келерові простори вперше вивчалися П. А. Широковим, які він назвав А-просторами. Потім ці простори вивчав Є. Келер. В літературі, як правило, ці простори називають келерові.

Задля зручності введемо в K_n операцію спряження [2]:

$$A_{\bar{i}\dots}^{\cdots} \equiv A_{\alpha\dots}^{\cdots} F_i^\alpha, \qquad B_{\dots}^{\bar{i}\dots} \equiv B_{\dots}^{\alpha\dots} F_\alpha^i.$$

Простором V_n *першого класу* називають гіперповерхню плаского простору. Його тензорні ознаки, необхідні та достатні умови мають вигляд

$$R_{hijk} = \epsilon(b_{hk} b_{ij} - b_{hj} b_{ik}), \qquad b_{ij,k} = b_{ik,j}, \tag{1}$$

тут $\epsilon = \pm 1$; $\quad b_{hi} = b_{ih}$. Згортаючи, отримаємо

$$R_{ij} = \epsilon(b b_{ij} - b_{\alpha j} b_i^\alpha), \tag{2}$$

де $b = b_{\alpha\beta} g^{\alpha\beta}$; $\quad b_j^i = b_{\alpha j} g^{\alpha i}$.

Домножимо (1) на b_m^h та згорнемо по h

$$b_m^\alpha R_{\alpha ijk} = \epsilon(b_m^\alpha b_{\alpha k} b_{ij} - b_m^\alpha b_{\alpha j} b_{ik}).$$

Після врахування (2) та (1), дістанемо

$$b_m^\alpha R_{\alpha ijk} = b R_{mijk} - R_{mk} b_{ij} + R_{mj} b_{ik}. \tag{3}$$

Подіємо операцією спряження по індексам j, k та віднімемо отримане від рівняння (3)

$$R_{mj} b_{ik} - R_{mk} b_{ij} - R_{m\bar{j}} b_{i\bar{k}} + R_{m\bar{k}} b_{i\bar{j}} = 0.$$

Згорнемо по індексам m, j :

$$R_{\alpha k} b_i^\alpha = \frac{R}{2} b_{ik}.$$

Для конформно-пласких келерових просторів першого класу доведено

Теорема 1. *Не існує конформно-пласких келерових просторів першого класу відмінних від пласких.*

Таким чином, клас конформно-пласких келерових просторів дорівнює двом.

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Відображення келерових просторів

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Аналітично планарною кривою L келерова простору називають криву, задану рівняннями $x^h = x^h(t)$ таку, що виконуються наступні умови:

$$\frac{d\xi^h}{dt} + \Gamma^h_{\alpha\beta}\xi^\alpha\xi^\beta = \rho_1(t)\xi^h + \rho_2(t)F^h_\alpha\xi^\alpha,$$

де $\xi^h \equiv \dfrac{dx^h}{dt}$, ρ_1, ρ_2 - функції аргументу t, Γ^h_{ij} — символи Христофеля K_n, а F^h_i його комплексна структура.

Дифеоморфізм γ між точками келерових просторів K_n і $\bar{K}_n$ називається голоморфно-проективним відображенням, якщо кожна аналітично планарна крива K_n переходить в аналітично планарну криву $\bar{K}_n$.

Якщо K_n допускає нетривіальне голоморфно-проективне відображення на $\bar{K}_n$, то в K_n існує розв'язок наступних рівнянь [1, 2]:

$$a_{ij,k} = \lambda_i g_{ik} + \lambda_j g_{ik} + \lambda_{\bar{i}} g_{\bar{j}k} + \lambda_{\bar{j}} g_{\bar{i}k}$$

відносно тензора a_{ij}, що задовольняє умовам

$$a_{ij} = a_{ji}; \qquad a_{\bar{i}\bar{j}} = a_{ij}, \qquad |a_{ij}| \neq 0$$

і ненульового вектора λ_i. Тут кома знак коваріантної похідної, а g_{ij} — метричний тензор.

Для вектора λ_i з необхідності виконуються умови:

$$\lambda_{i,\,j} = \lambda_{j,i} = \lambda_{\bar{i},\,\bar{j}}.$$

Тут застосована операція спряження: $A^{\dots}_{\bar{i}\dots} \equiv A^{\dots}_{\alpha\dots}F^\alpha_i$; $\qquad B^{\bar{i}\dots}_{\dots} \equiv B^{\alpha\dots}_{\dots}F^i_\alpha$ [3].

Розглянуто келерові простори K_n, тензор Річчі яких задовольняє умові

$$R_{ij,k} - R_{ik,j} = 0$$

або

$$R_{ij,k} + R_{jk,i} + R_{ki,j} = 0.$$

Для таких просторів доведено, що, якщо вони допускають нетривіальні голоморфно-проективні відображення, то вони є просторами сталої голоморфно-проективвної кривини.

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Метод растрової візуалізації перетинаючих геометричних тіл та побудови розгорток

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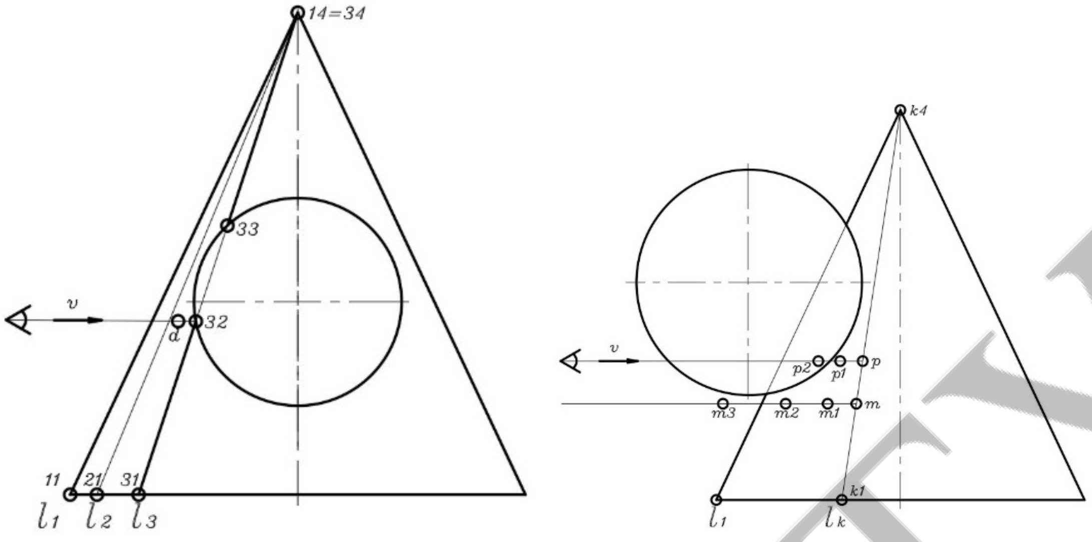
При візуалізації пересічних геометричних тіл виникають певні труднощі. По-перше, це знаходження лінії перетину, та візуалізація її деяких частин. По-друге, це візуалізація самих геометричних тіл. Практичні задачі побудови розгорток ускладняються при перетині декількох геометричних тіл. Пропоновані методи побудови аналогічні звичайним способам вирішення завдань із використанням знань із нарисної геометрії, тому громіздкі і трудомісткі.

Одним із методів описання геометричних тіл є застосування теорії R-функцій [1, 2]. Проведенні роботи в ХПІ показали перспективність застосування на практиці цих методів [3]. Широкому застосуванню цих методів на практиці заважає громіздкий аналітичний апарат, потребуючий описання геометричних тіл декількома рівняннями. Метою даної роботи є розроблення алгоритму спрощених схем візуалізації пересічних геометричних тіл та побудова їх розгорток.

Розглянемо перетин класичних конуса та циліндра (рис. 1А). Побудуємо растрове зображення з допомогою твірних, які утворюють одне тіло та перетинають інше. Конус як тверде геометричне тіло можливо виразити

$$R = R_1 + R_2 - |R_1 - R_2|\,.$$

Треба зауважити, що будь-яка точка, що розташовується всередині конуса має додатну функцію R. Точки які розташовані поза конусом мають від'ємну функцію R, а точки на конічній поверхні мають нульову функцію R. Можливо запрограмувати певну кількість твірних (рис. 1А), як на конічній (i) так і на циліндричній поверхні. Розглянемо твірні лінії, які утворюють конічну поверхню (рис.1Б). Для побудови точок на твірній лінії необхідно визначитися з її видимістю. Для цього в кожній початковій точці на твірній лінії, яка розташована на основі конуса будемо розраховувати нормаль до конічної поверхні. Це дозволяє визначити кут між нормаллю і вектором погляду спостерігача v. Якщо він гострий, то твірна невидима, якщо кут тупий, то твірна видима. Так твірні лінії конічної поверхні l_1, l_2, l_3 – видимі для спостерігача. Візьмемо твірну лінію l_3, яка перетинає циліндр. Пересуваючись із невеликим кроком по твірній, наприклад із точки на основі конуса (т. 31) до вершини (т. 34), будем



(а) (б)

Рис. 0.1. Перетин конуса з циліндром

розраховувати функцію R циліндра. При першому перетині циліндричної поверхні функція R змінить свій знак із від'ємного на додатній. При цьому координати попередньої точки занесемо у двовимірний масив (т. 32). Перший індекс це номер твірної, а другий точка входу. На виході з циліндру функція R також змінить свій знак із додатного на від'ємний. Координати точки занесемо до двовимірного масиву в якому другий індекс означає точку виходу (т. 33). Враховуючи непрозорість циліндра можливо зазначити, що всі точки твірної лінії між точками 32 та 33 невидимі.

Проблема в тому, що якусь частку видимої твірної може прикривати інше геометричне тіло. Визначити видимість точки видимої твірної на поверхні одного тіла, можливо розглянувши значення R-функції другого геометричного тіла при русі з цієї точки у зворотному напрямку вектора погляду спостерігача. Якщо R-функція зростає, то точка видима. Якщо R-функція зменшується, то треба пройти по напрямку вектора погляду спостерігача й тоді можливі два випадки. Для ілюстрації цих випадків розглянемо рис. 1Б в якому циліндр зміщений щодо вертикальної площини симетрії конуса. Розглянемо спочатку точку p на твірній l_3. Розрахуємо R-функцію в даній точці та перейдемо в точку p_1 де також визначимося зі значенням цієї функції. Порівнявши ці два значення можливо зробити висновок, що ми наближаємося до другого тіла (оскільки функція в першій точці за абсолютною величиною буде більше ніж у другій). Зробимо ще крок і опинимося в точці p_2. Точка на векторі потрапила в середину другого тіла (функція R змінить знак), що вказує на невидимість початкової точки p твірної лінії (друге тіло прикриває точку на видимій твірній першого тіла). Також можливий випадок коли точки на векторі погляду будуть наближатися до другого тіла, але пройдуть повз нього. Візьмемо точку m на цій твірній лінії. Для визначення її видимості пройдемося по вектору погляду у зворотному напрямку. Розрахунок R-функцій у точках m, m_1, m_2, дає змогу зробити висновок, що ми наближаємося до другого тіла, тому розрахунки R-функцій продовжуємо. Перейшовши в точку m_3 і порахувавши значення R-функції, яке за абсолютною величиною буде зростати можливо зробити заключення, що ми віддаляємося від другого тіла та встановити видимість точки на видимій твірній.

Знаючи кількість твірних на конічній поверхні легко розрахувати та побудувати сектор між двома суміжними твірними. Основна задача полягає в переносі точок перетину твірних ліній конуса з іншим геометричним тілом на твірні лінії розгортки. Твірні лінії конуса l_1, l_2 не мають точок перетину з циліндром (рис.1А). Це дає змогу легко побудувати перший сектор розгортки. На твірній лінії l_3 є точка входу в циліндр (т. 32) та точка виходу (т. 33). Знаючи координати цих двох точок, знайдемо довжину частини i–ої твірної від точки на основі (т. 31) до точки перетину з циліндром (т. 32), та довжину частини твірної від точки виходу з циліндра (т. 33) до вершини конуса (т. 34). Точність методу буде залежати від кількості твірних і від кроку між точками. Індексування точок входу твірних в інше геометричне тіло та їх виходу з тіла дає змогу об'єднати ці точки полілінією або сплайном.

Наведена методика побудови растрових 3-D моделей і розгорток лінійчатих поверхонь, що базується на теорії R-функцій. Вона дає змогу здійснити розрахунки і виконати реалістичні побудови 3-D моделей геометричних тіл, що перетинаються. З заданою точністю можливо побудувати розгортки геометричних тіл, що перетинаються, як із тонколистового так і товстостінного матеріалу.

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Ундулоїди та деякі їх деформації

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Ундулоїд- одна із поверхонь К. Делоне (C.Delaunay) [1], які застосовуються у газовій динаміці при дослідженні мильних плівок та бульбашок.

Пошук поля зміщення нескінченно малої (н.м.) деформації першого порядку зі стаціонарним тензором Річчі однозв'язної регулярної поверхні у E_3 просторі зводиться до дослідження та розв'язування диференціального рівняння другого порядку з частинними похідними відносно двох невідомих функцій $\mu(x^1, x^2)$ та $\varphi(x^1, x^2)$:

$$\rho^{\alpha\beta}\mu_{\alpha\beta} + s^{\alpha}\mu_{\alpha} = -K(d^{\alpha\beta}\varphi_{\alpha\beta} + l^{\alpha}\varphi_{\alpha} + 2H\varphi), \tag{1}$$

де $\rho^{\alpha\beta}$, $d^{\alpha\beta}$, s^{α}, l^{α}, K, H- відомі функції точки поверхні, $\mu_{\alpha\beta} = \frac{\partial^2\mu}{\partial x^{\alpha}\partial x^{\beta}}$, $\mu_{\alpha} = \frac{\partial\mu}{\partial x^{\alpha}}$.

Зокрема, якщо функція $\varphi(x^1, x^2)$ є певною характеристичною функцією (є розв'язком однорідного рівняння Вейнгартена [2]), то (1) буде диференціальним рівнянням гіперболічного типу відносно функції $\mu(x^1, x^2)$.

Доведено, що будь-яка поверхня класу C^5 ненульових гаусової та середньої кривин при певних граничних умовах допускає єдину н.м. деформацію зі стаціонарним тензором Річчі в класі C^2-поверхонь.

Слід зазначити, що рівняння (1) розглядалося у роботі [3] за умови $\mu(x^1, x^2) \in C^3$ є заздалегідь заданою функцією.

Для ундулоїда у лініях кривини за умови $\varphi(x^1, x^2) = 0$ рівняння (1) набуде вигляду:

$$\mu_{12} - \frac{2(2+\sin x^1)(4\sin^2 x^1 + 16\sin x^1 + 13)\cos x^1}{(1+2\sin x^1)(5+4\sin x^1)^2}\mu_2 = 0.$$

Скориставшись математичною системою MATHCAD для обчислення інтегралів при розв’язуванні цього рівняння, отримаємо наступний результат.

Ундулоїд допускає н.м. деформацію першого порядку зі стаціонарним тензором Річчі за умови, що функція $\varphi(x^1, x^2) = 0$. Тензорні поля при цьому представлені в явній формі та містять знайдену функцію

$$\mu(x^1, x^2) = \frac{1+2\sin x^1}{\sqrt[4]{5+\sin x^1}} e^{-\frac{3}{16(5+4\sin x^1)}} C(x^2),$$

де $C(x^2)$- довільна функція від однієї змінної. У випадку $\mu = 0$ ундулоїд буде жорстким.

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Канонічні F-планарні відображення

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Вивчались F-планарні відображення просторів афінної зв’язності, які були введені в розгляд Н. С. Сінюковим та Й. Мікешем [1]. Цей клас відображень є природним узагальненням геодезичних, голоморфно-проективних та квазігеодезичних відображень афіннозв’язних та ріманових просторів, наділених афінорними структурами.

F-планарні відображення просторів афінної зв’язності

$$f : (A_n, \Gamma^h_{ij}, F^h_i) \to (\overline{A}_n, \overline{\Gamma}^h_{ij}, F^h_i)$$

можуть бути двох типів: повні та канонічні. Нами розглянуто канонічний тип, який в загальній за відображенням системі координат (x_i) характеризується основними рівняннями:

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \phi_i(x)F^h_j(x) + \phi_j(x)F^h_i(x), \qquad h, i, j - 1, 2, ..., n,$$

де Γ_{ij}, $\overline{\Gamma}_{ij}^h$ - компоненти об'єктів зв'язності просторів A_n,$\overline{A}_n$ відповідно; ϕ_i - ковектор; F_i^h - афінор.

За означенням F-планарне відображення визначається лише на просторах з афінорною структурою F_i^h (в загальному випадку довільного типу). Ми досліджували спеціальний випадок, коли простір $A_n = V_n$, тобто є рімановим (V_n, g_{ij}, F_i^h), і афінор F_i^h задає на ньому келерову структуру еліптичного або гіперболічного типу:

$$F_\alpha^h F_i^\alpha = e\delta_i^h, \quad e = -1, +1,$$
$$F_{ij} + F_{ji} = 0, \qquad F_{ij} = g_{i\alpha} F_j^\alpha,$$
$$F_{i,j}^h = 0,$$

а $\overline{A}_n$ - локально плоский, тобто для його тензора Рімана маємо $\overline{R}_{ijk}^h = 0$. Тут , - знак коваріантної похідної в V_n.

Простори, які допускають F-планарне відображення на плоский простір $\overline{A}_n$, називають *F-плоскими*, а ті, що допускають канонічне F-планарне відображення на плоский простір, ми називаємо *канонічно F-плоскими*.

Ми довели, що тензор Рімана канонічно F-плоского простору має вид:

$$R_{ijk}^h = K\left(g_{hj}g_{ik} - g_{ij}g_{hk} - eF_{hj}F_{ik} + eF_{ij}F_{hk} - 2eF_{hi}F_{jk}\right),$$

$K = const$, тобто канонічно F-плоский простір необхідно є простором сталої голоморфної кривини. Метрики всіх таких просторів описані в [2]

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Table of contents