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- Differential geometry in the whole
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# On the matrix equation $AX - YB = C$ over Bezout domains

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Let  $R_{m,n}$  be the set of  $m \times n$  matrices over a commutative ring  $R$  with identity  $e \neq 0$ . Denote by  $I_n$  the identity  $n \times n$  matrix and by  $0_{m,n}$  the zero  $m \times n$  matrix. For any matrix  $A \in R_{m,n}$   $A^t$  denotes the transpose of  $A$ . We will denote by  $GL(m, R)$  the set of invertible matrices in  $R_{m,m}$ . We will write  $C_{\downarrow i}$  for the  $i$ -th column of the matrix  $C \in R_{m,n}$  and  $\text{vec}(C)$  will denote an ordered stock of columns of  $C$ , i.e.,

$$\text{vec}(C) = \begin{bmatrix} C_{\downarrow 1} \\ C_{\downarrow 2} \\ \vdots \\ C_{\downarrow n} \end{bmatrix}.$$

In this note we present alternative methods for finding solutions of the Sylvester matrix equation

$$AX - YB = C, \quad (1)$$

where  $A, B$  and  $C$  are given matrices of suitable sizes over a commutative domain.

This equation has been considered by several authors including Roth [12] over a field, Hartwig [6] over a regular ring, Gustafson [5] over a commutative ring with identity, Emre and Silverman [3] over a polynomial ring, Özgüler [7] over a principal ideal domain, Dajić [2] over an associative ring with unit. In general, Gustafson [5] has proved that equation (1) over a commutative ring  $R$  with identity has a solution  $(X, Y)$  over  $R$  if and only if the matrices  $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$  and  $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$  are equivalent. This is a generalization of Roth's result [12], which gives the same criterion for the case, where  $R$  is a field. Similar considerations on solvability of equation (1) can be found in original paper [1].

**1.** Let  $R$  be a Bezout domain. Without reducing the generality we will assume that  $A \in R_{m,m}$ ,  $B \in R_{n,n}$  and  $C \in R_{m,n}$ , and  $X, Y$  are unknown  $m \times n$  matrices over  $R$ . Using the Kronecker product matrix equation (1) may be considered in the form of equivalent linear system (see [4])

$$(I_n \otimes A) \text{vec}(X) - (B^t \otimes I_m) \text{vec}(Y) = \text{vec}(C).$$

**Theorem 1.** *Matrix equation (1) over Bezout domain  $R$  is solvable if and only if matrices*

$$\begin{bmatrix} (I_n \otimes A) & (B^t \otimes I_m) & 0_{mn,1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} (I_n \otimes A) & (B^t \otimes I_m) & \text{vec}(C) \end{bmatrix}$$

*are column equivalent, i.e., the right Hermite normal forms of these matrices are the same.*

**Corollary 2.** *Let  $A_i \in R_{m,m}$ ,  $B_i \in R_{n,n}$  and  $C_i \in R_{m,n}$ ,  $i = 1, 2$ . Matrix equations  $A_1X - YB_1 = C_1$  and  $A_2X - YB_2 = C_2$  have a common solution over Bezout domain  $R$  if and only if matrices*

$$\begin{bmatrix} (I_n \otimes A_1) & (B_1^t \otimes I_m) & 0_{mn,1} \\ (I_n \otimes A_2) & (B_2^t \otimes I_m) & 0_{mn,1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} (I_n \otimes A_1) & (B_1^t \otimes I_m) & \text{vec}(C_1) \\ (I_n \otimes A_2) & (B_2^t \otimes I_m) & \text{vec}(C_2) \end{bmatrix}$$

*are column equivalent, i.e., the right Hermite normal forms of these matrices are the same.*

We were using results of papers [9] and [10] for proving Theorem 1.

**2.** In this parch  $R$  is a principal ideal domain. We denote by  $(a, b)$  the greatest common divisor of nonzero elements  $a, b \in R$ . Let  $A \in R_{m,m}$  and  $\text{rank } A = r$ . For the matrix  $A$  there exist matrices  $U, V \in GL(m, R)$  such that  $UAV = S_A = \text{diag}(a_1, a_2, \dots, a_r, 0, \dots, 0)$  is the Smith normal form of  $A$ .

**Theorem 3.** Let  $A \in R_{m,m}$ ,  $B \in R_{n,n}$ ,  $C \in R_{m,n}$  and  $\text{rank} A = p$ ,  $\text{rank} B = q$ . Further, let  $U_A, V_A \in GL(m, R)$  and  $U_B, V_B \in GL(n, R)$  such that

$$U_A A V_A = S_A = \text{diag}(a_1, a_2, \dots, a_p, 0, \dots, 0), \quad U_B B V_B = S_B = \text{diag}(b_1, b_2, \dots, b_q, 0, \dots, 0).$$

Matrix equation (1) is solvable over  $R$  if and only if

$$U_A C V_B = \left[ \begin{array}{ccc|c} f_{11} & \cdots & f_{1q} & 0_{p,n-q} \\ \vdots & \ddots & \vdots & \\ f_{p1} & \cdots & f_{pq} & \\ \hline 0_{m-p,q} & & & 0_{m-p,n-q} \end{array} \right] \quad \text{and} \quad (a_i, b_j) | f_{ij} \text{ (divides)}$$

for all  $i = 1, 2, \dots, p$   $i j = 1, 2, \dots, q$ .

It is clear that if matrices  $A \in R_{m,m}$  and  $B \in R_{n,n}$  are nonsingular and  $(\det A, \det B) = e$ , then matrix equation (1) is solvable for an arbitrary matrix  $C \in R_{m,n}$ .

Suppose that matrix equation (1) is solvable under the conditions of Theorem 3. Then for invariant factors  $a_i$  and  $b_j$  of matrices  $A$  and  $B$  respectively there exist  $\alpha_{ij}, \beta_{ij} \in R$  such that  $a_i \alpha_{ij} - \beta_{ij} b_j = f_{ij}$  for all  $i = 1, 2, \dots, p$  and  $j = 1, 2, \dots, q$ . Put

$$X_\alpha = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1q} \\ \vdots & \ddots & \vdots \\ \alpha_{p1} & \cdots & \alpha_{pq} \end{bmatrix} \quad \text{and} \quad Y_\beta = \begin{bmatrix} \beta_{11} & \cdots & \beta_{1q} \\ \vdots & \ddots & \vdots \\ \beta_{p1} & \cdots & \beta_{pq} \end{bmatrix}.$$

Then for arbitrary matrices  $P_{12}, Q_{12} \in R_{p,n-q}$ ;  $P_{21}, Q_{21} \in R_{m-p,q}$  and  $P_{22}, Q_{22} \in R_{m-p,n-q}$  the pair of matrices

$$X_P = V_A^{-1} \begin{bmatrix} X_\alpha & P_{12} \\ P_{21} & P_{22} \end{bmatrix} V_B^{-1} \quad \text{and} \quad Y_Q = U_A^{-1} \begin{bmatrix} Y_\beta & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} U_B^{-1}$$

is the general solution of matrix equation (1). We note that Theorem 3 can be used for finding solutions with some properties of equation (1) (see [8] and [11]).

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