

International scientific conference
**«Algebraic and geometric
methods of analysis»**

Book of abstracts



May 30 - June 4, 2018,
Odesa,
Ukraine

<https://www.imath.kiev.ua/~topology/conf/agma2018>

The Lie-algebraic structure of the Lax-Sato integrable superanalogs for the Liouville heavenly type equations

Oksana Ye. Hentosh

(Pidstryhach Inst. for Applied Problems of Mech. and Math., NASU, Lviv, Ukraine)

E-mail: ohen@ukr.net

Yarema A. Prykarpatsky

(Institute of Mathematics, NASU, Kyiv, Ukraine)

E-mail: yarpry@gmail.com

In the paper [1] the general Lie-algebraic approach to constructing the Lax-Sato integrable heavenly type systems has been developed. It is based on the classical Adler-Kostant-Symes (AKS) theory and \mathcal{R} -operator structures related with the loop Lie algebra $\widetilde{diff}(\mathbb{T}^n)$ of the vector fields on the n -dimensional torus \mathbb{T}^n and adjacent Lie algebra $diff_{hol}(\mathbb{C} \times \mathbb{T}^n) \subset diff(\mathbb{C} \times \mathbb{T}^n)$ of the holomorphic in the ‘‘spectral’’ parameter $\lambda \in \mathbb{S}_{\pm}^1$ vector fields on $\mathbb{C} \times \mathbb{T}^n$. A generalization of this Lie-algebraic scheme, related with the loop Lie algebra $\widetilde{diff}(\mathbb{T}^{1|N})$ of superconformal vector fields on the $1|N$ -dimensional supertorus $\mathbb{T}^{1|N} \simeq \mathbb{S}^1 \times \Lambda_1^N$, where $\Lambda := \Lambda_0 \oplus \Lambda_1$ is an infinite-dimensional Grassmann algebra over $\mathbb{C} \subset \Lambda_0$, has been proposed in [2] for $n = 1$ and applied to construct the Lax-Sato integrable superanalogs of the Mikhalev-Pavlov heavenly equation for every $N \in \mathbb{N} \setminus \{4; 5\}$. In our report the Lax-Sato integrable superanalogs of the Liouville heavenly type equations are obtained by use the loop Lie algebra $\widetilde{diff}(\mathbb{T}_\mathbb{C}^{1|N})$ of the superconformal vector fields on $\mathbb{T}_\mathbb{C}^{1|N} \simeq \mathbb{T}_\mathbb{C}^1 \times \Lambda_1^N$ as a result of some diffeomorphic mapping in the space of variables $(z, \vartheta) \in \mathbb{T}_\mathbb{C}^{1|N}$, where $\vartheta := (\vartheta_1, \dots, \vartheta_N)^\top$, $\vartheta_i \in \Lambda_1$, $i = \overline{1, N}$.

At first one introduces the superderivatives $D_{\vartheta_i} := \partial/\partial\vartheta_i + \vartheta_i\partial/\partial z$, $z \in \mathbb{T}_\mathbb{C}^1$, $\vartheta_i \in \Lambda_1$, $i = \overline{1, N}$, in the superspace $\Lambda_0 \times \Lambda_1^N$. The loop Lie algebra $\widetilde{diff}(\mathbb{T}_\mathbb{C}^{1|N})$ are formed by the superconformal vector fields such as $\tilde{a} := a\partial/\partial z + \langle Da, D \rangle / 2$, where $D := (D_{\vartheta_1}, D_{\vartheta_2}, \dots, D_{\vartheta_N})^\top$, $\vartheta := (\vartheta_1, \dots, \vartheta_N)^\top$, $a \in C^\infty(\mathbb{T}_\mathbb{C}^{1|N}; \Lambda_0)$, with the commutator

$$[\tilde{a}, \tilde{b}] := \tilde{c} = c\partial/\partial z + \langle Dc, D \rangle / 2, \quad c = a\partial b/\partial z - b\partial a/\partial z + \langle Da, Db \rangle / 2,$$

This loop Lie algebra $\widetilde{diff}(\mathbb{T}_\mathbb{C}^{1|N})$ allows the splitting $\widetilde{diff}(\mathbb{T}_\mathbb{C}^{1|N}) = \widetilde{diff}(\mathbb{T}_\mathbb{C}^{1|N})_+ \oplus \widetilde{diff}(\mathbb{T}_\mathbb{C}^{1|N})_-$. Here the Lie subalgebras $\widetilde{diff}(\mathbb{T}_\mathbb{C}^{1|N})_{\pm}$ are assumed to be formed by the vector fields $\tilde{a}(z)$ on $\mathbb{T}_\mathbb{C}^{1|N}$, being holomorphic in $z \in \mathbb{S}_{\pm}^1 \subset \mathbb{C}$ respectively, where $\tilde{a}(\infty) = 0$ for any $\tilde{a}(z) \in \widetilde{diff}(\mathbb{T}_\mathbb{C}^{1|N})_-$.

The nontrivial Casimir invariant $h^{(p_y)} \in I(\widetilde{diff}(\mathbb{T}_\mathbb{C}^{1|N})^*)$ on a dense subspace $\widetilde{diff}(\mathbb{T}_\mathbb{C}^{1|N})^* \simeq \Lambda^1(\mathbb{T}_\mathbb{C}^1)$ of the dual space through the pairing $(\tilde{l}, \tilde{a}) := \text{res}_{\lambda \in \mathbb{C}} \int_{\mathbb{S}^1} z^{-1} dz \int_{\Lambda_1^N} (la) d^N \vartheta$, $\tilde{l} := ldz \in \widetilde{diff}(\mathbb{T}_\mathbb{C}^{1|N})^*$, satisfies the relationship

$$(l(\nabla h^{(p_y)}(l))^2)_z - Nl((\nabla h^{(p_y)}(l))^2)_z/4 = (-1)^N \langle Dl, D(\nabla h^{(p_y)}(l))^2 \rangle / 4, \quad (1)$$

where $\nabla h^{(p_y)}(\tilde{l}) := \nabla h^{(p_y)}(l)\partial/\partial z + \langle D\nabla h^{(p_y)}(l), D \rangle / 2$. If the corresponding gradient has the asymptotic expansion $\nabla h^{(p_y)}(l) \simeq \sum_{j \leq r} V_j z^j$, where $p_y = r$ and $V_j \in C^2(\mathbb{R}^2 \times \Lambda_1^N; \Lambda_0)$, $j \in \mathbb{Z}$, $j \leq r$, $r \in \mathbb{Z}_+$, are some functional parameters, as $|z| \rightarrow \infty$, we can construct the Hamiltonian flow

$$dl/dy = -l_z \nabla h_+^{(p_y)}(l) - (4 - N)l(\nabla h_+^{(p_y)}(l))_z/2 + (-1)^N \langle Dl, D\nabla h_+^{(p_y)}(l) \rangle / 2 \quad (2)$$

in the framework of the classical AKS-theory. The constant Casimir invariant $h^{(p_t)} \in I(\widetilde{diff}(\mathbb{T}_\mathbb{C}^{1|N})^*)$ generates the trivial flow

$$dl/dt = 0. \quad (3)$$

The compatibility condition of these two flows for all $y, t \in \mathbb{R}$ is equivalent to the following system of two *a priori* compatible linear vector field equations

$$\partial\psi/\partial y + V\partial\psi/\partial z + \langle DV, D\psi \rangle / 2 = 0, \quad \partial\psi/\partial t = 0, \quad (4)$$

where $\nabla h_+^{(p_y)}(l) := V$, $V = V(y, t, \vartheta; z) = \sum_{0 \leq j \leq r} V_j z^j$, and $\nabla h^{(p_t)}(l) = 0$, for a smooth function $\psi \in C^2(\mathbb{R}^2 \times \Lambda_1^N; \Lambda_0)$. In this case we have the evolutions

$$dz/dy = V - \langle \theta, DV \rangle / 2, \quad d\vartheta/dy = (DV)/2, \quad dz/dt = 0, \quad d\theta/dt = 0. \quad (5)$$

Under the diffeomorphic mapping $z \mapsto z - \kappa - \langle \theta, \eta \rangle := \lambda$ and $\vartheta \mapsto \vartheta + \eta := \tilde{\vartheta}$, $\eta := (\eta_1, \dots, \eta_N)^\top$, $\tilde{\vartheta} := (\tilde{\vartheta}_1, \dots, \tilde{\vartheta}_N)^\top$, on $\mathbb{T}_{\mathbb{C}}^{1|N}$, generated by the functions $\kappa := \kappa(y, t) \in C^3(\mathbb{R}^2; \Lambda_0)$ and $\eta := \eta(y, t) \in C^3(\mathbb{R}^2; \Lambda_1^N)$, the equations (4) are rewritten as

$$\partial\psi/\partial y + W\partial\psi/\partial\lambda + \langle \tilde{D}W, \tilde{D}\psi \rangle / 2 = 0, \quad \partial\psi/\partial t - U\partial\psi/\partial\lambda - \langle \tilde{D}U, \tilde{D}\psi \rangle / 2 = 0, \quad (6)$$

where $W := W(y, t, \tilde{\vartheta}; \lambda) = \sum_{0 \leq j \leq r} W_j \lambda^j$, $U := U(y, t, \tilde{\vartheta})$, $\tilde{D} := (D_{\vartheta_1}, D_{\vartheta_2}, \dots, D_{\vartheta_N})^\top$ and $D_{\tilde{\vartheta}_i} := \partial/\partial\tilde{\vartheta}_i + \tilde{\vartheta}_i \partial/\partial\lambda$, $i = \overline{1, N}$. Taking into account the evolutions (5) and

$$d\lambda/dy = W - \langle \tilde{\theta}, \tilde{D}W \rangle / 2, \quad d\tilde{\vartheta}/dy = (\tilde{D}W)/2, \quad d\lambda/dt = -U + \langle \tilde{\theta}, \tilde{D}U \rangle / 2, \quad d\tilde{\vartheta}/dt = -(\tilde{D}U)/2,$$

one obtains the function W such as $W = \tilde{V} + \langle \eta, \tilde{D}\tilde{V} \rangle - \partial\kappa/\partial y + \langle \eta, \partial\eta/\partial y \rangle$, where $\tilde{V} := \tilde{V}(y, t, \tilde{\vartheta}; \lambda) = V(y, t, \vartheta; z)|_{z=\lambda+\kappa+\langle\theta,\eta\rangle, \vartheta=\tilde{\vartheta}-\eta}$. Furthermore, the superderivatives transform by the rules $D_{\vartheta_i} = D_{\tilde{\vartheta}_i} - 2\eta_i \partial/\partial\lambda$, $i = \overline{1, N}$, and the functions κ and η obey the relationships $\partial\kappa/\partial t - \langle \eta, \partial\eta/\partial t \rangle = U$, $\partial\eta/\partial t = -(\tilde{D}U)/2$.

If $W_2 := 1$ and $U := 1/2 \exp \varphi$, $\varphi := \varphi(y, t, \vartheta)$, the compatibility condition for the first order partial differential equations (6) leads to the Lax-Sato integrable superanalogs of Liouville heavenly type equations [3]

$$\varphi_{yt} = \exp \varphi - \sum_{i=1}^N (\partial\varphi_y/\partial\tilde{\vartheta}_i)(\partial \exp \varphi/\partial\tilde{\vartheta}_i)/4, \quad W_0 := 1, \quad (7)$$

$$\varphi_{yt} - \varphi_{tt} = \exp \varphi - \sum_{i=1}^N (\partial(\varphi_y - \varphi_t)/\partial\tilde{\vartheta}_i)(\partial \exp \varphi/\partial\tilde{\vartheta}_i)/4, \quad W_0 := -1/2 \exp \varphi. \quad (8)$$

Because of the relationship (1) the element $\tilde{l} \in \widetilde{diff}(\mathbb{T}_{\mathbb{C}}^{1|N})^*$ can be found explicitly. For example, in the case of $r = 2$ and $N = 1$ it has the following form

$$\tilde{l}(y, t, \vartheta_1; z) = (z^{-4}(\vartheta_1(1 - 2v_1 z^{-1} + (3v_1^2 - 2v_0)z^{-2}) + \beta_1/2 + (\beta_0/4 - 9\beta_1 v_1/8)z^{-1}))dz, \quad (9)$$

where $V_2 := 1$ and $V_j := v_j + \vartheta_1 \beta_j$, $j = \overline{0, 1}$. Thus, one can formulate the following proposition.

Proposition 1. *For all $N \in \mathbb{N}$ the super-Liouville heavenly type equations (7) and (8) possess the Lax-Sato vector field representations (6), being equivalent to the commutability condition of two Hamiltonian flows (2) and (3) on $\widetilde{diff}(\mathbb{T}_{\mathbb{C}}^{1|N})^*$. In the case of $N = 1$ the equations (7) and (8) are put into the AKS-scheme for the loop Lie algebra $\widetilde{diff}(\mathbb{T}_{\mathbb{C}}^{1|N})$ with the element $\tilde{l} \in \widetilde{diff}(\mathbb{T}_{\mathbb{C}}^{1|N})^*$ in the form (9).*

REFERENCES

- [1] Oksana E. Hentosh, Yarema A. Prykarpatsky, Denis Blackmore, Anatolij Prykarpatski. Lie-algebraic structure of Lax-Sato integrable heavenly equations and the Lagrange-d'Alembert principle. *J. of Geometry and Physics.*, 120 : 208-227, 2017.
- [2] Oksana Ye. Hentosh, Yarema A. Prykarpatsky. The Lie-algebraic structure of integrable "heavenly" superflows. *Proceedings of 18th Intern. Scientific Mykhailo Kravchuk Conference, October 7-10, 2017, Kyiv*, volume 1. Kyiv: NTUU "KPI", 2017. (in Ukrainian)
- [3] Leonid V. Bogdanov, Boris G. Konopelchenko. Grassmannians $Gr(N-1, N+1)$, closed differential $N-1$ forms and N -dimensional integrable systems. *J. Phys. A: Math. Theor.*, 46(8), 085201: 26 pp., 2012.

Зміст

N. Aygor, H. Burhanzade <i>Secondary school students' misconceptions about linear algebra</i>	3
S. Bardyla, H. Kvasnytsia <i>Semitopological graph inverse semigroups</i>	4
B. A. Bhayo <i>On inequalities of generalized elliptic integrals</i>	5
Bodzioch M., Choiński M., Foryś U. <i>A criss-cross model of tuberculosis for heterogenous population</i>	6
Bolotov D. V. <i>Foliations with leaves of non-positive curvature and bounded total curvature on closed 3-manifolds</i>	7
E. Bonacci <i>Algebraic and geometric questions about a 6D physics</i>	9
F. Bulnes <i>Mukai-Fourier Transform in Derived Categories to Solutions of the Field Equations: Gravitational Waves as Oscillations in the Space-Time Curvature/Spin IV</i>	10
H. Burhanzade, N. Aygor <i>A study on the teaching methods in determinants</i>	12
Damla Yaman <i>Order continuity properties of lattice ordered algebras</i>	13
Denega I. <i>Problem on non-overlapping polycylindrical domains with poles on the boundary of a polydisk</i>	14
A. Dudko, V. Pivovarchik <i>Inverse three spectra problem for a Stieltjes string with the Neumann boundary conditions</i>	16
Eftekharinasab K. <i>On the existence of a global diffeomorphism between Fréchet spaces</i>	18
Glazunov N. <i>Class groups of rings with divisor theory, L-functions and moduli spaces</i>	19
O. Gok <i>b-bimorphisms</i>	21
Gül E. <i>On the second regularized trace formula for a differential operator with unbounded coefficients</i>	22
Hentosh O. Ye., Prykaratsky Ya. A. <i>The Lie-algebraic structure of the Lax-Sato integrable superanalogs for the Liouville heavenly type equations</i>	24
V. Herasymov <i>In a natural topological sense a typical linear nonhomogeneous differential equation in the ring $Z[[x]]$ has no solutions from $Z[[x]]$.</i>	26
Juraev D. A. <i>On the Cauchy problem for matrix factorizations of the Helmholtz equation</i>	27
M. E. Kansu <i>Macroscopic electromagnetism via complex quaternions</i>	29
Vladimir V. Kisil <i>An extension of Möbius–Lie geometry with conformal ensembles of cycles</i>	30
Konovenko N., Lychagin V. <i>Rational differential invariants for oriented primary visual cortex</i>	32