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Algebraic and Geometric Methods of Analysis

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LIST OF TOPICS

- Algebraic methods in geometry
- Differential geometry in the large
- Geometry and topology of differentiable manifolds
- General and algebraic topology
- Dynamical systems and their applications
- Geometric problems in mathematical analysis
- Geometric and topological methods in natural sciences

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ІНТЕРНАЦІОНАЛЬНИЙ ЦЕНТР СПІВРОБІТНИЦТВА

Mixed volumes/areas and distribution of zeros of holomorphic functions

Bulat N. Khabibullin

(Bashkir State University, Ufa, Bashkortostan, Russian Federation)

E-mail: khabib-bulat@mail.ru

Roman R. Muryasov

(Bashkir State University, Ufa, Bashkortostan, Russian Federation)

E-mail: romrumur@yandex.ru

We denote by $\mathbb{N} := \{1, 2, \dots\}$, \mathbb{R} , $\mathbb{R}^+ := \{x \in \mathbb{R} : x \geq 0\}$, $\mathbb{R}_*^+ := \mathbb{R}^+ \setminus \{0\}$, and \mathbb{C} the sets of *natural*, of *real*, of *positive*, of *strictly positive*, and of *complex* numbers, each endowed with its natural order (\leq , \sup / \inf), algebraic, geometric and topological structure.

Mixed areas/volumes. Let S be a bounded subset in \mathbb{C} with the *support function* [1, Ch. 1], [2]

$$\text{Sp}_S(z) := \sup_{z \in \mathbb{C}} \sup_{w \in S} \text{Re}(z\bar{w}), \quad \text{sp}_S(t) := \text{Sp}_S(e^{it}), \quad \text{and} \quad \Delta_S(t) := (\text{sp}_S)'_{\text{left}}(t) + \int_0^t \text{sp}_S(x) dx,$$

where $(\text{sp}_S)'_{\text{left}}$ is the left derivative of sp_S . The *mixed area (of Minkowski)* $F(S_1, S_2)$ of bounded sets $S_1, S_2 \subset \mathbb{C}$ is integrals [1, Ch. 1, 3], [2], [3, § 4]

$$F(S_1, S_2) := \frac{1}{2} \int_0^{2\pi} \text{sp}_{S_1}(t) d\Delta_{S_2}(t) = \frac{1}{2} \int_0^{2\pi} \left(\text{sp}_{S_1} \text{sp}_{S_2} - (\text{sp}_{S_1})'_{\text{left}} (\text{sp}_{S_2})'_{\text{left}} \right)(t) dt = F(S_2, S_1).$$

Convexity with respect to a pair of functions. Let $I \subset \mathbb{R}$ be an open interval, and let $f_1: I \rightarrow \mathbb{R}$ and $f_2: I \rightarrow \mathbb{R}$ be a pair of functions. A function $g: I \rightarrow \mathbb{R}$ will be called *convex with respect to the pair* f_1, f_2 , or, briefly, (f_1, f_2) -*convex* if there is a number $d > 0$ such that for each $x_1, x_2 \in I$ with $|x_1 - x_2| < d$ and for each $C_1, C_2 \in \mathbb{R}$ such that

$$g(x_1) \leq C_1 f_1(x_1) + C_2 f_2(x_1), \quad g(x_2) \leq C_1 f_1(x_2) + C_2 f_2(x_2),$$

we have $g(x) \leq C_1 f_1(x) + C_2 f_2(x)$ for each $x \in (x_1, x_2)$ [4, Ch. I, § 1]. So, if $g: \mathbb{R}_*^+ \rightarrow \mathbb{R}_*^+$ is (f_1, f_2) -convex for the pair $f_1: x \mapsto x$ and $f_2: x \mapsto 1/x$, then we say that g is $(x, 1/x)$ -convex.

Entire functions in \mathbb{C} . Even the following special result develops [3, § 4], [5], [6, Ch. 3, 4.2].

Let $f \neq 0$ be an entire function of exponential type with the indicator of growth of f denoted by

$$\text{Ind}_1[f](z) := \limsup_{0 < r \rightarrow +\infty} \frac{\ln |f(rz)|}{r} \in \mathbb{R}, \quad z \in \mathbb{C}.$$

The function $\text{Ind}_1[f]$ is convex and positive homogeneous on \mathbb{C} . Therefore, there is a non-empty convex compact set $I_f \subset \mathbb{C}$ with $\text{Sp}_{I_f} = \text{Ind}_1[f]$ called the *indicator diagram* of this entire function f .

Theorem 1. *Let $f \neq 0$ be an entire function of exponential type with the indicator diagram I_f . Suppose that the function f vanish on a sequence $Z = (z_k)_{k \in \mathbb{N}} \subset \mathbb{C}$, i.e., $f(z_k) = 0$ for each $k \in \mathbb{N}$. If K is a non-empty convex compact subset in \mathbb{C} , and $g: \mathbb{R}_*^+ \rightarrow \mathbb{R}_*^+$ is an increasing $(x, 1/x)$ -convex function on \mathbb{R}_*^+ , such that*

$$0 < \liminf_{0 < x \rightarrow +\infty} \frac{g(x)}{x} \leq \limsup_{0 < x \rightarrow +\infty} \frac{g(x)}{x} < +\infty, \quad (1)$$

then

$$\limsup_{1 < a \rightarrow +\infty} \limsup_{r \rightarrow +\infty} \frac{\pi}{\int_r^{ar} g(1/x) dx} \sum_{r < |z_k| \leq ar} g\left(\frac{1}{|z_k|}\right) \text{sp}_K\left(\frac{z_k}{|z_k|}\right) \leq F(I_f, K). \quad (2)$$

Besides, for the identity function $g: x \mapsto x$ and for any non-empty convex compact subsets S and K in \mathbb{C} , there is an entire function $f \neq 0$ of exponential type with zero sequence $Z = (z_k)_{k \in \mathbb{N}} \subset \mathbb{C}$ and the indicator diagram $I_f = S$ such that we have the equality in (2).

Theorem 1 can be extended to entire functions of finite order $\rho \in \mathbb{R}^+$ of one or several complex variables with significant generalizations of mixed areas/volumes for ρ -convex sets. Thus, we obtain numerous exact results on the completeness of systems of entire functions in classical function spaces on sets in \mathbb{C}^n for $n \in \mathbb{N}$ in terms of the *mutual indicator* of entire function and set [5], [6, Ch. 3, 4.2].

Here we note only the simplest version of the application of Theorem 1 to completeness questions.

Completeness of exponential systems. For a compact subset S of \mathbb{C} , we denote by $C(S) \cap \text{Hol}(\text{int}S)$ the normed space of all continuous functions $f: S \rightarrow \mathbb{C}$ such that the restriction f to the interior $\text{int}S$ of S is holomorphic if this interior $\text{int}S$ is non-empty, equipped with the norm $\|f\|_S := \sup_{s \in S} |f(s)|$.

Theorem 2. *Let S be a compact subset of the complex plane such that $\mathbb{C} \setminus S$ is connected. Let $Z = (z_k)_{k \in \mathbb{N}} \subset \mathbb{C}$ be a sequence of pairwise distinct numbers. If there are a non-empty compact convex subset $K \subset \mathbb{C}$ and an increasing $(x, 1/x)$ -convex function $g: \mathbb{R}_*^+ \rightarrow \mathbb{R}_*^+$ satisfying (1) such that*

$$\limsup_{1 < a \rightarrow +\infty} \limsup_{r \rightarrow +\infty} \frac{\pi}{\int_r^{ar} g(1/x) dx} \sum_{r < |z_k| \leq ar} g\left(\frac{1}{|z_k|}\right) \text{sp}_K\left(\frac{\bar{z}_k}{|z_k|}\right) > F(I, K).$$

then the closure of the linear hull of $\{e^{z_k s} : k \in \mathbb{N}\}$ in $C(S) \cap \text{Hol}(\text{int}S)$ coincides with $C(S) \cap \text{Hol}(\text{int}S)$.

Holomorphic functions in the unit disk/ball. In [7] and [8], we first used ρ -trigonometrically convex and ρ -subspherical functions to study zero sets of holomorphic functions on the unit disk in \mathbb{C} and on the unit ball in \mathbb{C}^n , respectively. Some of these results can be obtained in a more general form in terms of mixed areas/volumes and Hausdorff measures of zero sets.

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