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Algebraic and Geometric Methods of Analysis

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Dvytro Grave
(25.08.1863 - 19.12.1939)
Academician of the Ukrainian Academy of Sciences, the first director of the Institute of Mathematics of NAS of Ukraine

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List of topics

- Algebraic methods in geometry
- Differential geometry in the large
- Geometry and topology of differentiable manifolds
- General and algebraic topology
- Dynamical systems and their applications
- Geometric and topological methods in natural sciences
- Geometric problems in mathematical analysis

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Hopf-Rinow theorem of sub-Finslerian geometry

Layth M. Alabdulsada
(Dep. of Math., College of Science, University of Al-Qadisiyah, Al-Qadisiyah, 58001, Iraq)
E-mail: layth.muhsin@qu.edu.iq

Laszlo Kozma
(Inst. of Math., University of Debrecen, H-4002 Debrecen, P.O. Box 400, Hungary)
E-mail: kozma@unideb.hu

The sub-Finslerian geometry means that the metric F is defined only on a given subbundle of the tangent bundle, called a horizontal bundle. In the paper, a version of the Hopf-Rinow theorem is proved in the case of sub-Finslerian manifolds, which relates the properties of completeness, geodesically completeness, and compactness. The sub-Finsler bundle, the exponential map and the Legendre transformation are deeply involved in this investigation.

We construct a sub-Finsler bundle, which plays a major role in the formalization of the sub-Hamiltonian in sub-Finsler geometry. Moreover, the sub-Finsler bundle allows an orthonormal frame for the sub-Finsler structure. We introduce the notion of an exponential map in sub-Finsler geometry. At the end, our main theorem is stated and proved.

Theorem 1. *Let $(M, \mathcal{D}, F)$ be any connected sub-Finsler manifold, where $\mathcal{D}$ is bracket generating distribution. The following conditions are equivalent:*

(i) *The metric space (M, d) is forward complete.*
(ii) *The sub-Finsler manifold $(M, \mathcal{D}, F)$ is forward geodesically complete.*
(iii) *$\Omega^*_x = \mathcal{D}^*_x$, additionally, the exponential map is onto if there are no strictly abnormal minimizers.*
(iv) *Every closed and forward bounded subset of (M, d) is compact.*

Furthermore, for any $x, y \in M$ there exists a minimizing geodesic γ joining x to y, i.e. the length of this geodesic is equal to the distance between these points.

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Geometric properties of interception curves

Yagub Aliyev

(School of IT and Engineering, ADA University, Ahmadbey Aghaoglu str. 61, Baku AZ1008, Azerbaijan)

E-mail: yaliyev@ada.edu.az

In this study, a plane curve, which was named as *Interception Curve*, was discussed. This curve can be defined in the following way. Suppose one point moves with constant velocity along a straight line, and another point, at the beginning one unit apart from the line and the first point on this line, moves with the same constant speed so that it always stays on a line passing through the first point and the initial position of the second point. This plane curve appears in problems related to the interception of high-speed targets by beam rider missiles (hence the name *Interception Curve*) [2, 5]. This curve was also mentioned in [4, 6, 1]. In [3], at Sect. 1.460 and Sect. 1.507), some methods based on polar and Cartesian coordinates were proposed to find an explicit representation for this curve.

Problem 1. If two points $P(x, y)$ and Q, initially at $O(0, 0)$ and $A(1, 0)$, respectively, move uniformly so that Q is on the line $x = 1$, and P is on the ray OQ then what curve does the point P draw?

Answer. Let us use polar coordinates $r = |OP|$ and $\angle AOQ = \theta$. We obtain ordniary differential equation

$$r(\theta)^2 + (r'(\theta))^2 = \frac{1}{\cos^4\theta}, \tag{1}$$

with initial condition $r(0) = 0$. Note that in the cartesian coordinates, (1) can be written as

$$x^2\sqrt{1 + (y'(x))^2} = y'x - y, \tag{2}$$

with initial condition $y(0) = 0$. By solving this equation, we obtain the parametrization (cf. [3], Sect. 1.507, where the roles of x and y are interchanged)

$$\begin{cases} x(p) = \frac{1}{\sqrt{p}}\int_1^p \frac{\sqrt{t}dt}{2\sqrt{t^2-1}}, \\ y(p) = \frac{\sqrt{p^2-1}}{\sqrt{p}}\int_1^p \frac{\sqrt{t}dt}{2\sqrt{t^2-1}} - \left(\int_1^p \frac{\sqrt{t}dt}{2\sqrt{t^2-1}}\right)^2 \ (p \geq 1). \end{cases} \tag{3}$$

Using all these, the following results are obtained:

Theorem 2. *Suppose that U is the y intercept of the tangent line of the curve (3) at the point P, and this tangent line intersects the line $x = 1$ at point and T. Then*

(1) $x \cdot |UP| = |OU|$,

(2) $\sin\angle QPT = \frac{x^2}{|OP|} = \frac{x}{|OQ|}$,

where x is the abscissa of the point $P(x, y)$.

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