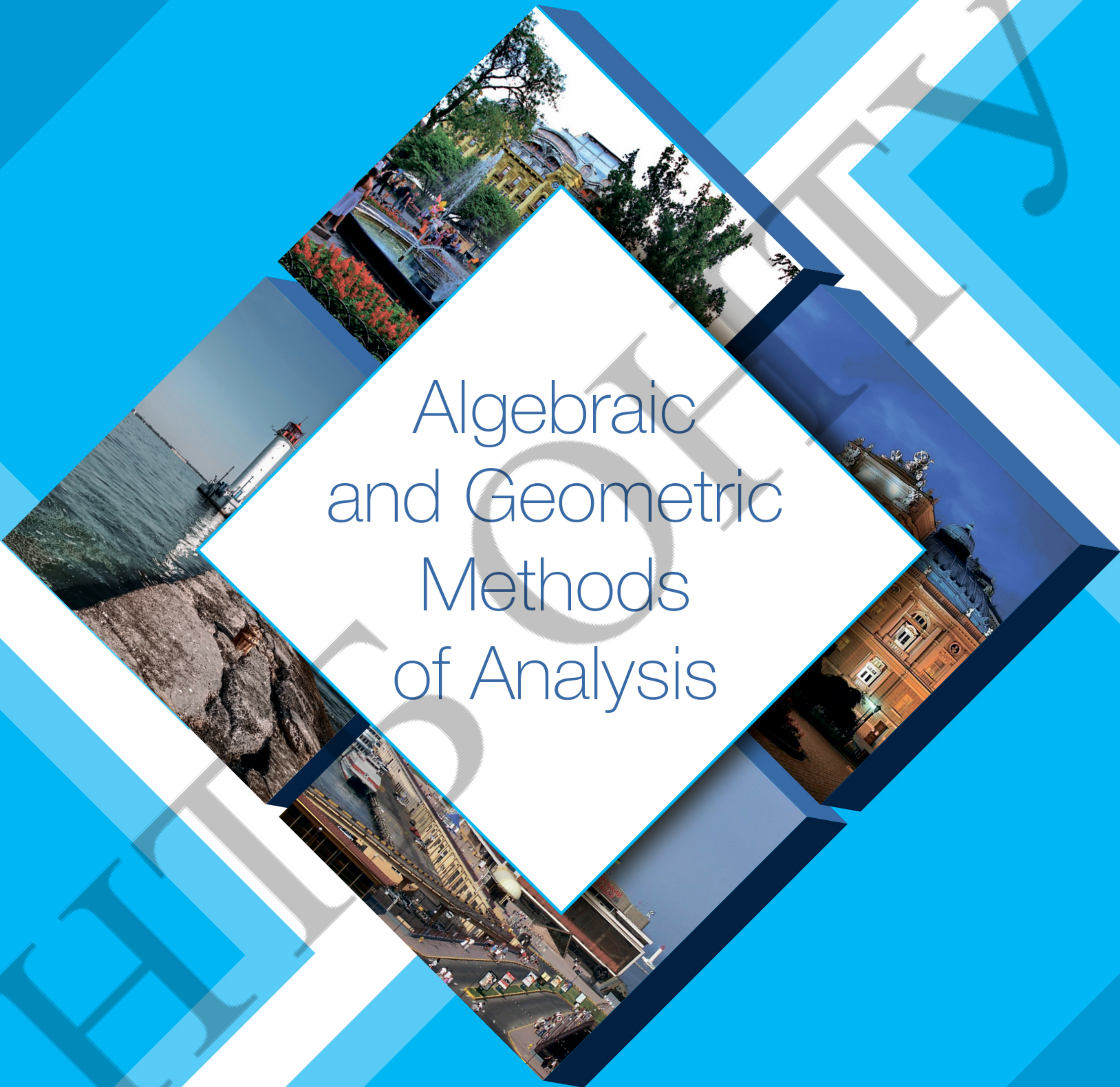
International
Scientific Conference

Algebraic and Geometric Methods of Analysis



27-30 May 2024

Odesa, Ukraine

The purpose of this conference is to bring together researchers in geometry, topology, algebra, analysis and dynamical systems and to provide for them a forum to present their recent work to colleagues from different nationalities. This way we aim to stimulate discussion about the latest findings in geometrical and topological methods in analysis and to increase international collaboration.

The conference continues the traditional annual conference «Geometry in Odesa» holding from 2004, and hosted by Odesa National University of Technology (Odesa National Academy of Food Technologies till 2021). From 2017 the conference was renamed to «Algebraic and geometric methods of analysis» (AGMA).

The Conference languages: Ukrainian and English.

List of topics

- Algebraic methods in geometry
- Differential geometry in the large
- Geometry and topology of differentiable manifolds
- General and algebraic topology
- Dynamical systems and their applications
- Geometric and topological methods in natural sciences
- Geometric problems in mathematical analysis

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Octonionic Stiefel manifolds and vector bundles

Marek Golasiński
(UWM in Olsztyn, Poland)
E-mail: marekg@matman.uwm.edu.pl

Francisco Gómez Ruiz
(Universidad de Málaga, España)
E-mail: gomez_ruiz@uma.es

The octonions $\mathbb{O}$ satisfy a weaker form of associativity. Namely, they are alternative and power associative only and are not as well known as complex numbers $\mathbb{C}$ and the quaternions $\mathbb{H}$ which are much more widely studied and used.

This talk studies: Stiefel and Grassmann varieties, and vector bundles over octonions $\mathbb{O}$.

Stiefel manifolds. Let $\mathbb{F}$ stand for $\mathbb{R}$, the reals, $\mathbb{C}$, the complex numbers or $\mathbb{H}$, the quaternions.

Recall that the Stiefel manifold $V_{n,r}(\mathbb{F})$ for $r \leq n$ is the set of all orthonormal r-frames in $\mathbb{F}^n$ which can be thought of as a set of $n \times k$ matrices by writing a r-frame as a matrix of k column vectors in $\mathbb{F}^n$. We then have

$$V_{n,r}(\mathbb{F}) = \left\{A \in M_{n,r}(\mathbb{F}) : \bar{A}^t A = I_r\right\}$$

and define

$$V_{n,r}(\mathbb{O}) = \left\{A \in M_{n,r}(\mathbb{O}) : \bar{A}^t A = I_r\right\}.$$

Those yield $V_{n,r}(\mathbb{F})$ and $V_{n,r}(\mathbb{O})$ as algebraic varieties over $\mathbb{R}$ (see [1, 3] for details).

Each $V_{n,r}(\mathbb{F})$ can be viewed as a homogeneous space:

$$V_{n,r}(\mathbb{F}) \cong \mathrm{U}(n,\mathbb{F})/\mathrm{U}(n-r,\mathbb{F}).$$

But, for $V_{n,r}(\mathbb{O})$ we have:

Proposition 1. (1) *$V_{n,r}(\mathbb{O})$ is a compact smooth submanifold of $M_{n,r}(\mathbb{O})$ for any $r \leq n$.*

(2) *$V_{n,r}(\mathbb{O})$ is path-connected for any $r \leq n$ and the map $\pi : V_{n,r+1}(\mathbb{O}) \to V_{n,r}(\mathbb{O})$, given by $\pi(A|v) = A$, is a smooth fibre bundle.*

Grassmann manifolds. Grassmann manifold $G_{n,r}(\mathbb{F})$ is a differentiable manifold that parameterizes the set of all r-dimensional linear subspaces of $\mathbb{F}^n$. Since the rank of an orthogonal projection operator equals its trace, we can identify

$$G_{n,r}(\mathbb{F}) = \left\{A \in M_n(\mathbb{F}) : A = \bar{A}^t = A^2, \operatorname{tr}(A) = r\right\}$$

and define

$$G_{n,r}(\mathbb{O}) = \left\{A \in M_n(\mathbb{O}) : A = \bar{A}^t = A^2, \operatorname{tr}(A) = r\right\}.$$

Those yield $G_{n,r}(\mathbb{F})$ and $G_{n,r}(\mathbb{O})$ as algebraic varieties over $\mathbb{R}$ (see [1, 3] for details).

Each $G_{n,r}(\mathbb{F})$ can be viewed as a homogeneous space:

$$G_{n,r}(\mathbb{F}) \cong \mathrm{U}(n,\mathbb{F})/\mathrm{U}(r,\mathbb{F}) \times \mathrm{U}(n-r,\mathbb{F}).$$

Furthermore, we have the principal $\mathrm{U}(r)$-bundle

$$\mathrm{U}(r,\mathbb{F}) \hookrightarrow V_{n,r}(\mathbb{F}) \to G_{n,r}(\mathbb{F})$$

for the Stiefel map $V_{n,r}(\mathbb{F}) \to G_{n,r}(\mathbb{F})$.

Due to the non-associativity of $\mathbb{O}$ we do not have a Stiefel map $\pi : V_{n,r}(\mathbb{O}) \to G_{n,r}(\mathbb{O})$, but we may define a subset $V'_{n,r}(\mathbb{O}) \subseteq V_{n,r}(\mathbb{O})$ as follows: $A \in V'_{n,r}(\mathbb{O})$ if the set of all entries of A

generate an associative subalgebra of $\mathbb{O}$. Then, we have a Stiefel map $\pi : V'_{n,r}(\mathbb{O}) \to G_{n,r}(\mathbb{O})$ given by $\pi(A) = A\bar{A}^t$.

Similarly to $V'_{n,r}(\mathbb{O})$, we define $G'_{n,r}(\mathbb{O})$ as follows: $A \in G'_{n,r}(\mathbb{O})$ if all entries of A generate an associative subalgebra of $\mathbb{O}$. It is clear that the Stiefel map $\pi : V'_{n,r}(\mathbb{O}) \to G_{n,r}(\mathbb{O})$ yields a surjective map $\pi : V'_{n,r}(\mathbb{O}) \to G'_{n,r}(\mathbb{O})$. In particular, $G'_{n,r}(\mathbb{O})$ is piecewise-smooth path-connected.

Vector bundles

By analogy with real, complex or quaterionic vector bundles, define an octonionic vector bundle of rank r over some space X as a continuous map

$$f : X \to G'_{-,r}(\mathbb{O}).$$

The following result holds by adapting proof of [1, Theorem 12.1.7] or [2, Theorem 2.5]:

Theorem 2. *Any smooth map $X \to G'_{n,r}(\mathbb{O})$ with $r \leq n$ and X, a compact non-singular algebraic set, is homotopic to an entire rational map from X to $G'_{n,r}(\mathbb{O})$.*

Then, we derive:

Corollary 3. *Any smooth map $X \to G_{n,r}(\mathbb{O})$ with $n = 2, 3$ and $r = 1$ is homotopic to an entire rational map $X \to G_{n,r}(\mathbb{O})$. In particular, any continuous map $\mathbb{S}^n \to \mathbb{S}^8$ is homotopic to an entire rational map $\mathbb{S}^n \to \mathbb{S}^8$.*

References

[1] J. Bochnak, M. Coste, and M.F. Roy, "Real Algebraic Geometry", Ergeb. Math. **36**, Springer-Verlag, Berlin, Heidelberg, New York (1998).

[2] J. Bochnak, W. Kucharz, *Algebraic approximation of mappings into spheres*, Michigan Math. J. **34** (1987), 119-125.

[3] M. Golasiński and F. Gómez Ruiz, "Grassmann and Stiefel varieties over composition algebras", Springer-Verlag, RSME Springer Series **9**, (2023).

Some series involving central binomial coefficients

Taras Goy

(Vasyl Stefanyk Precaprathian National University)

E-mail: taras.goy@pnu.edu.ua

Using Maclaurin expansion $\arcsin = \sum_{n=0}^{\infty} \frac{1}{2^{2n}}\binom{2n}{n}\frac{x^{2n+1}}{2n+1}$ and, for non-zero real variable x, formulas

$$\Re\Big(\arcsin\left(x\sqrt{i}\,\right)\Big) = \arctan\sqrt{\frac{\sqrt{1+x^4}-1}{x^2}},$$

$$\Im\Big(\arcsin\left(x\sqrt{i}\,\right)\Big) = \operatorname{arctanh}\sqrt{\frac{\sqrt{1+x^4}-1}{x^2}},$$

we obtain some series involving central binomial coefficients $\binom{2n}{n}$; see [1] for more details.

Theorem 1. *For $|x| \leq 1$, we have*

$$\sum_{n=0}^{\infty} \frac{(-1)^{\lfloor 3n/2 \rfloor}}{4^n(2n+1)}\binom{2n}{n}x^{2n+1} = \sqrt{2}\arctan\left(\frac{\sqrt{\sqrt{1+x^4}-1}}{x}\right),$$