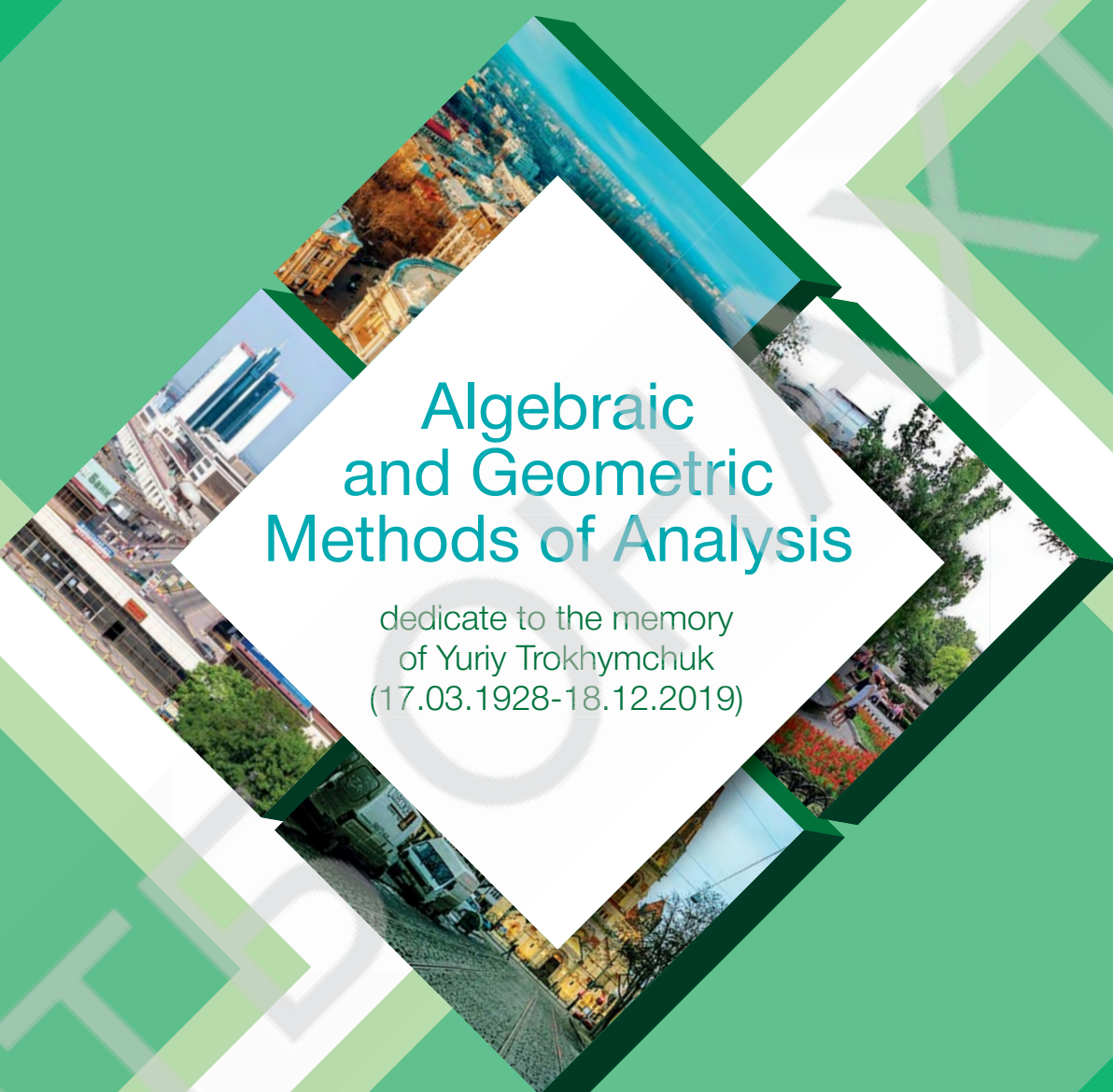


International
Online Conference



**Algebraic
and Geometric
Methods of Analysis**

dedicate to the memory
of Yuriy Trokhymchuk
(17.03.1928-18.12.2019)

May 25-28, 2021
Odesa, Ukraine

LIST OF TOPICS

- Topological methods in analysis
- Geometric problems of complex and mathematical analysis
- Algebraic methods in geometry
- Differential geometry in the whole
- Geometry and topology of differentiable manifolds
- General and algebraic topology
- Geometric and topological methods in natural sciences

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IW contractions and their generalizations

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Contractions of Lie algebras are a kind of limit processes between orbits of such algebras. In 1953, Inönü and Wigner studied special contractions of Lie algebras as a part of broader study of contractions of Lie groups and their representations. These contractions were generalized by Doebner and Melsheimer in 1967. A rigorous general definition of contractions of Lie algebras was given by Saletan in 1961. He also studied contraction whose matrices are first-order polynomials with respect to contraction parameters. Since then, a number of conjectures about various ways of realizing contractions of Lie algebras had accumulated in the literature.

Definition 1. Let V be an n -dimensional vector space over $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$, $n < \infty$, and let $\mathcal{L}_n = \mathcal{L}_n(\mathbb{F})$ denote the set of all possible Lie brackets on V . Given $\mu \in \mathcal{L}_n$ and $U \in C((0, 1], \text{GL}(V))$, define the family of $\mu_\varepsilon \in \mathcal{L}_n$, $\varepsilon \in (0, 1]$, by $\mu_\varepsilon(x, y) := U_\varepsilon^{-1}\mu(U_\varepsilon x, U_\varepsilon y) \forall x, y \in V$. If for any $x, y \in V$ there exists the limit $\lim_{\varepsilon \rightarrow +0} \mu_\varepsilon(x, y) =: \mu_0(x, y)$, then $\mathfrak{g}_0 = (V, \mu_0)$ is a well-defined Lie algebra and is called a *contraction* of the Lie algebra $\mathfrak{g} = (V, \mu)$. The procedure $\mathfrak{g} \rightarrow \mathfrak{g}_0$ providing \mathfrak{g}_0 from \mathfrak{g} is also called a *contraction*. If a basis of V is fixed, the parameter matrix $U_\varepsilon = U(\varepsilon)$, $\varepsilon \in (0, 1]$, is called the *contraction matrix* of the contraction $\mathfrak{g} \rightarrow \mathfrak{g}_0$.

Definition 2. The contraction $\mathfrak{g} \rightarrow \mathfrak{g}_0$ is called a *Inönü–Wigner (IW) contraction* if its matrix U_ε can be represented in the form $U_\varepsilon = AW_\varepsilon P$, where the matrices A and P are nonsingular and constant (i.e., they do not depend on ε) and $W_\varepsilon = \text{diag}(\varepsilon^{\alpha_1}, \dots, \varepsilon^{\alpha_n})$ for some $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. The n -tuple of exponents $(\alpha_1, \dots, \alpha_n)$ is called the *signature* of the generalized IW-contraction $\mathfrak{g} \rightarrow \mathfrak{g}_0$. A *simple IW-contraction* is a generalized IW-contraction with signature consisting of zeros and ones.

The following assertion, which stood as a conjecture for a long time, was proved in [5].

Theorem 3. *Any generalized IW-contraction is equivalent to a generalized IW-contraction with an integer signature (and the same associated constant matrices).*

One of these conjectures was that any contraction of Lie algebras can be realized as a generalized IW-contraction. This is true for contractions between three-dimensional real or complex Lie algebras. Consider four-dimensional real Lie algebras defined, up to antisymmetry of Lie bracket, by the following nonzero commutation relations:

$$\begin{aligned} 2A_{2,1}: & \quad [e_1, e_2] = e_1, [e_3, e_4] = e_3; \\ A_1 \oplus A_{3,2}: & \quad [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3; \\ A_{4,1}: & \quad [e_2, e_4] = e_1, [e_3, e_4] = e_2; \\ A_{4,10}: & \quad [e_1, e_3] = e_1, [e_2, e_3] = e_2, [e_1, e_4] = -e_2, [e_2, e_4] = e_1. \end{aligned}$$

Hereafter we use the Mubarakzhanov's nomenclature for low-dimensional Lie algebras, and \mathfrak{g}_{\dots} denotes the complexification of the algebra A_{\dots} . All contractions of four-dimensional real Lie algebras were realized in [1, 2] via generalized IW-contractions except two contractions, $2A_{2,1} \rightarrow A_1 \oplus A_{3,2}$ and $A_{4,10} \rightarrow A_1 \oplus A_{3,2}$. Since the complexifications of the algebras $2A_{2,1}$ and $A_{4,10}$ are isomorphic, there was only one exception for the complex case, $2\mathfrak{g}_{2,1} \rightarrow \mathfrak{g}_1 \oplus \mathfrak{g}_{3,2}$.

Theorem 4 ([4]). (i) *There exists a unique contraction between four-dimensional complex Lie algebras, $2\mathfrak{g}_{2,1} \rightarrow \mathfrak{g}_1 \oplus \mathfrak{g}_{3,2}$, that is not equivalent to a generalized IW-contraction.*

(ii) *Precisely two contractions between four-dimensional real Lie algebras, $2A_{2,1} \rightarrow A_1 \oplus A_{3,2}$ and $A_{4,10} \rightarrow A_1 \oplus A_{3,2}$, cannot be realized as generalized IW-contractions.*

Combining the results of [1, 2, 4] also yields the following assertion.

Theorem 5 ([4]). *Any generalized IW-contraction between four-dimensional complex (resp. real) Lie algebras is equivalent to one with parameter exponents in $\{0, 1, 2, 3\}$. The exponents in $\{0, 1, 2\}$ suffice for all such contractions except $2A_{2,1} \rightarrow A_{4,1}$, $A_{4,10} \rightarrow A_{4,1}$ and $\mathfrak{so}(3) \oplus A_1 \rightarrow A_{4,1}$ in the real case and $2\mathfrak{g}_{2,1} \rightarrow \mathfrak{g}_{4,1}$ in the complex case, where the minimal tuple of exponents is $(3, 2, 1, 1)$.*

Definition 6. The contraction $\mathfrak{g} \rightarrow \mathfrak{g}_0$ is called *diagonal* if its matrix U_ε can be represented in the form $U_\varepsilon = AW_\varepsilon P$, where A and P are constant nonsingular matrices and $W_\varepsilon = \text{diag}(f_1(\varepsilon), \dots, f_n(\varepsilon))$ for some continuous functions $f_i: (0, 1] \rightarrow \mathbb{F} \setminus \{0\}$.

Theorem 7 ([5]). *Any diagonal contraction is equivalent to a generalized IW-contraction with an integer signature.*

Consider the n -dimensional ($n \geq 5$) solvable real Lie algebras $\mathfrak{a} := A_{5,38} \oplus (n-5)A_1$ and $\mathfrak{a}_0 := A_{2,1} \oplus A_{2,1} \oplus (n-4)A_1$ whose nonzero commutation relations are exhausted, up to antisymmetry of Lie bracket, by the following:

$$\mathfrak{a}: \quad [e_1, e_3] = e_3, \quad [e_2, e_4] = e_4, \quad [e_1, e_2] = e_5, \quad \mathfrak{a}_0: \quad [e_1, e_3] = e_3, \quad [e_2, e_4] = e_4.$$

Theorem 8 ([3]). *The Euclidean norm of any contraction matrix that realizes the contraction of the algebra \mathfrak{a} to the algebra \mathfrak{a}_0 approaches infinity at the limit point. The same is true for the complex counterpart of this contraction.*

Definition 9. A realization of a contraction with a matrix-function that is linear in the contraction parameter is called a *Saletan (linear) contraction*.

Theorem 10 ([6]). *Up replacing the algebras \mathfrak{g} and \mathfrak{g}_0 with isomorphic ones, every Saletan contraction $\mathfrak{g} \rightarrow \mathfrak{g}_0$ is realized by a matrix of the canonical form*

$$E^{n_0} \oplus J_\varepsilon^{n_1} \oplus \dots \oplus J_\varepsilon^{n_s}, \quad \text{or, equivalently,} \quad E^{n_0} \oplus J_0^{n_1} \oplus \dots \oplus J_0^{n_s} + \varepsilon E^n,$$

where $n_0 + \dots + n_s = n$, E^m is the $m \times m$ identity matrix, and J_λ^m denotes the $m \times m$ Jordan block with an eigenvalue λ .

Hence any Saletan contraction can be realized by a matrix of the form $AS_\varepsilon B$, where A and B are constant nonsingular matrices and the matrix-valued function S_ε is in the above canonical form. The tuple $(n_0; n_1, \dots, n_s)$, where n_1, \dots, n_s constitute a partition of the dimension $n - n_0$ of the Fitting null component relative to U_0 and $n_0 \in \{0, \dots, n\}$, is called the *signature* of this Saletan contraction.

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