

International
Scientific Conference



Algebraic and Geometric Methods of Analysis



Devoted to 160 anniversary of
Dvytro Grave
(25.08.1863 - 19.12.1939)
Academician of the Ukrainian
Academy of Sciences, the
first director of the Institute of
Mathematics of NAS of Ukraine

May 29 – June 1, 2023
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LIST OF TOPICS

- Algebraic methods in geometry
- Differential geometry in the large
- Geometry and topology of differentiable manifolds
- General and algebraic topology
- Dynamical systems and their applications
- Geometric and topological methods in natural sciences
- Geometric problems in mathematical analysis

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The Riemann-Hilbert problem and holomorphic bundles framed along a real hypersurface

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The Riemann sphere $\mathbb{P}_{\mathbb{C}}^1 = \mathbb{C} \cup \{\infty\}$ decomposes as the union $\mathbb{P}_{\mathbb{C}}^1 = \bar{U}^- \cup \bar{U}^+$ of two closed disks $\bar{U}^- = \bar{D}$, $\bar{U}^+ = \mathbb{P}^1 \setminus D$ intersecting along their boundary $\partial\bar{U}^{\pm} = S^1$. The Riemann-Hilbert problem, as stated by Hilbert in [2, Kapitel X], asks:

The Riemann-Hilbert Problem. Let $\Gamma : S^1 \rightarrow \mathrm{GL}(r, \mathbb{C})$ be a smooth map. Find the pairs (Y^-, Y^+) of continuous maps $Y^{\pm} : \bar{U}^{\pm} \rightarrow \mathbb{C}^r$ which are holomorphic on U^{\pm} and satisfy the condition $Y^+|_{S^1} = \Gamma Y^-|_{S^1}$.

More generally, consider

- (1) A representation $\rho : G \rightarrow \mathrm{GL}(V)$ of a complex Lie group G on a finitely dimensional complex vector space V ,
- (2) A map $\Gamma : S \rightarrow G$ of class \mathcal{C}^{κ} with $\kappa \in [0, \infty]$,
- (3) An integer $m \in \mathbb{Z}$ and a V -valued polynomial $\gamma \in V[z]$. Put $d := \deg(\gamma) \in \mathbb{Z}_{\geq -1}$.

Regarding ∞ as an effective divisor on $\mathbb{P}_{\mathbb{C}}^1$, γ can be interpreted as an element of $H^0(\mathcal{O}(d\infty)_{(d+1)\infty} \otimes V)$. We ask:

The general RH problem on \mathbb{P}^1 . Find the space of pairs (Y^-, Y^+) of continuous maps

$$Y^- : \bar{U}^- \rightarrow V, \quad Y^+ : \bar{U}^+ \setminus \{\infty\} \rightarrow V$$

with Y^- holomorphic on U^- , Y^+ holomorphic on $U^+ \setminus \{\infty\}$ such that $Y_S^+ = \rho(\Gamma)Y_S^-$, and

$$\lim_{z \rightarrow \infty} (z^{d-m} Y^+(z) - \gamma(z)) = 0$$

The geometric interpretation of the latter condition: Y^+ extends as a section of the sheaf $\mathcal{O}(m\infty) \otimes_{\mathbb{C}} V$ on U^+ whose image in $H^0(\mathcal{O}(m\infty)_{(d+1)\infty} \otimes V)$ via the obvious morphism is $z^{m-d} \otimes \gamma$. Hilbert's original problem is obtained taking ρ to be the canonical representation of $\mathrm{GL}(r, \mathbb{C})$ on \mathbb{C}^r , $m = 0$, and $\gamma = 0$.

Complex geometric point of view: Consider the sheaf \mathcal{V}^{Γ} of local solutions of the RH problem with $m = \gamma = 0$; this sheaf is given explicitly by:

$$W \mapsto \left\{ \begin{array}{c} \left(\begin{array}{c} f^- \\ f^+ \end{array} \right) \in \begin{array}{c} \mathcal{C}^0(W \cap \bar{U}^-, V) \\ \times \\ \mathcal{C}^0(W \cap \bar{U}^+, V) \end{array} \left| \begin{array}{l} f^+|_{W \cap S} = \rho(\Gamma) f^-|_{W \cap S}, \\ f^{\pm} \text{ is holomorphic on } W \cap U^{\pm} \end{array} \right. \end{array} \right\}.$$

Theorem 1. Suppose $\kappa \in (1, \infty] \setminus \mathbb{N}$. The sheaf of $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}$ -modules \mathcal{V}^{Γ} is locally free of rank $\dim(V)$ and coincides with the apparently smaller sheaf

$$\left\{ \begin{array}{c} \left(\begin{array}{c} f^- \\ f^+ \end{array} \right) \in \begin{array}{c} \mathcal{C}^{\kappa}(W \cap \bar{U}^-, V) \\ \times \\ \mathcal{C}^{\kappa}(W \cap \bar{U}^+, V) \end{array} \left| \begin{array}{l} f^+|_{W \cap S} = \rho(\Gamma) f^-|_{W \cap S}, \\ f^{\pm} \text{ is holomorphic on } W \cap U^{\pm} \end{array} \right. \end{array} \right\}.$$

We have an obvious identification

$$H^0(\mathcal{O}(d\infty)_{(d+1)\infty} \otimes V) \xrightarrow{\cong} H^0(\mathcal{V}^{\Gamma}(d\infty)_{(d+1)\infty}),$$

so γ gives an element $\nu_\gamma^\Gamma \in H^0(\mathcal{V}^\Gamma(d\infty)_{(d+1)\infty})$.

Consider the short exact sequence of coherent sheaves on $\mathbb{P}_\mathbb{C}^1$

$$0 \rightarrow \mathcal{V}^\Gamma((m-d-1)\infty) \rightarrow \mathcal{V}^\Gamma(m\infty) \rightarrow \mathcal{V}^\Gamma(m\infty)_{(d+1)\infty} \rightarrow 0$$

and the associated cohomology long exact sequence.

- Corollary 2.** (1) *The space of solutions of the general RH problem is non-empty if and only if the image of $z^{m-d} \otimes \nu_\gamma^\Gamma$ via the connecting morphism $H^0(\mathcal{V}^\Gamma(m\infty)_{(d+1)\infty}) \rightarrow H^1(\mathcal{V}^\Gamma((m-d-1)\infty))$ vanishes.*
- (2) *If this is the case, this space has the natural structure of an affine space with model space $H^0(\mathcal{V}^\Gamma((m-d-1)\infty))$, and can be identified with the pre-image of $z^{m-d} \otimes \nu_\gamma^\Gamma$ via the morphism $H^0(\mathcal{V}^\Gamma(m\infty)) \rightarrow H^0(\mathcal{V}^\Gamma(m\infty)_{(d+1)\infty})$.*
- (3) *(Regularity) Any solution (Y^-, Y^+) of a RH problem with Γ of class \mathcal{C}^κ is also of class \mathcal{C}^κ up to the boundary.*

Note that, by Grothendieck's classification theorem, \mathcal{V}^Γ splits as a direct sum of invertible sheaves, so $\mathcal{V}^\Gamma \simeq \bigoplus_{j=1}^r \mathcal{O}(n_j)$ with $n_j \in \mathbb{Z}$ and $\sum_{j=1}^r n_j = \deg(\mathcal{V}^\Gamma)$. For the canonical representation of $\mathrm{GL}(r, \mathbb{C})$ on \mathbb{C}^r we have $\deg(\mathcal{V}^\Gamma) = -\deg(\det(\Gamma))$.

The RH problem can be naturally generalized in the framework of Riemann surfaces as follows: we replace $\mathbb{P}_\mathbb{C}^1$ by an arbitrary closed Riemann surface X , the circle S^1 by an arbitrary (not necessarily connected, not necessarily separating) oriented closed curve $S \subset X$, and Γ by a map $\Gamma \in \mathcal{C}^\kappa(S, G)$. Let \widehat{X}_S be the Riemann surface with boundary obtained by cutting X along S . The unknown of the RH problem associated with these data is a meromorphic map $Y : \widehat{X}_S \setminus \partial\widehat{X}_S \dashrightarrow V$, which extends continuously around $\partial\widehat{X}_S$ and satisfies a compatibility condition associated with Γ . In this general framework one also has a complex geometric interpretation of the space of solutions which generalizes Corollary 2.

All these results are applications of a general gluing theorem for holomorphic bundles. The same theorem can be used to prove an isomorphism theorem between moduli spaces of framed holomorphic bundles [5], [1]:

Let E be a differentiable vector bundle of rank r on a closed complex manifold X , $S \subset X$ a closed, separating real hypersurface, $X = \bar{X}^- \cup \bar{X}^+$ the corresponding decomposition of X as union of compact complex manifolds with boundary, and $E^\pm := E|_{\bar{X}^\pm}$.

Theorem 3. *The moduli space $\mathcal{M}_S(E)$ of S -framed holomorphic structures on E can be identified with the fibre product of the moduli spaces $\mathcal{M}_{\partial\bar{X}^\pm}(E^\pm)$ of boundary framed (formally) holomorphic structures on E^\pm over the space of Cauchy-Riemann operators on the trivial bundle of rank r on S .*

Note that $\mathcal{M}_{\partial\bar{X}^\pm}(E^\pm)$ can be further identified with moduli spaces of boundary framed Hermitian-Einstein connections on E^\pm using a version of the classical Kobayashi-Hitchin correspondence [3] for complex manifolds with boundary [1], [6].

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About some Steiner trees

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will talk about the famous Steiner problem.

We denote by \mathcal{H}^1 the linear Haurdorff measure (roughly speaking, length).

Problem 1 (Euclidean Steiner problem). Let C be a compact subset of \mathbb{R}^d . To find a closed S such that $S \cup C$ is connected and $\mathcal{H}^1(S)$ is minimal.

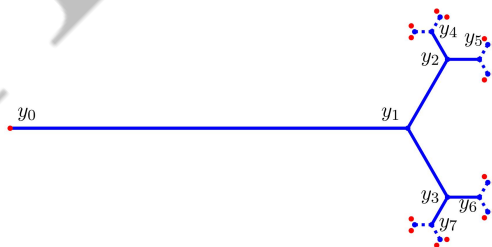
Some properties (if $\mathcal{H}^1(S) < \infty$) hold:

- S exists;
- S contains no loops;
- only two variants of neighbourhoods for points from $S \setminus C$;
- only two variants of a neighbourhood of a point $x \in S \setminus C$:
 - a regular tripod (x is a **branching point**);
 - a segment; x is an inner point.
- S contains at most countable number of branching points.

Usually S is usually called **Steiner tree**, and it is called **indecomposable** (irreducible, full), when $S \setminus C$ is connected. If C is totally disconnected then S should be connected.

Theorem 2 (Paolini–Stepanov–T, 2015; Cherkashin–T. 2023; Paolini–Stepanov 2023). *There is a compact planar set C for which the unique solution of the Steiner problem is indecomposable and has infinite number of triple points.*

In the theorem C and Σ are self-similar fractals with a sufficiently small scale factor.



Theorem 3 (Basok, Cherkashin, T., 2022). *In the plane for $m \geq 4$ the set of m terminals (considered as a subset of \mathbb{R}^{2m}) with non unique Steiner trees has the Hausdorff dimension $2m - 1$.*

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