



International
Scientific Conference



Algebraic and Geometric Methods of Analysis



Devoted to 160 anniversary of
Dvytro Grave
(25.08.1863 - 19.12.1939)
Academician of the Ukrainian
Academy of Sciences, the
first director of the Institute of
Mathematics of NAS of Ukraine

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LIST OF TOPICS

- Algebraic methods in geometry
- Differential geometry in the large
- Geometry and topology of differentiable manifolds
- General and algebraic topology
- Dynamical systems and their applications
- Geometric and topological methods in natural sciences
- Geometric problems in mathematical analysis

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Fermat–Torricelli sets of finite sets of points in Euclidean plane

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Definition 1 ([8]). Let (X, ρ) be a metric space and $x_1, \dots, x_n \in X$ be a finite collection of points in X . A point $\bar{x} \in X$ is called a *Fermat–Torricelli point* for x_1, \dots, x_n whenever for each $x \in X$ the following inequality holds true:

$$\sum_{k=1}^n \rho(\bar{x}, x_k) \leq \sum_{k=1}^n \rho(x, x_k).$$

Definition 2. *Fermat–Torricelli set* for fixed points $\{x_1, \dots, x_n\}$ is a set of all Fermat–Torricelli points for this collection of points.

In the case when $X = \mathbb{R}^n$ is the Euclidean space with the standard metric, then for every finite collection of points $x_1, \dots, x_n \in \mathbb{R}^n$ the set of its Fermat–Torricelli points is non-empty, convex and compact. The problem of finding the Fermat–Torricelli set is called the *Fermat–Torricelli problem*.

This problem has both geometric and probabilistic interpretation. We can describe discrete probabilistic space $\Omega = \{x_1, \dots, x_n\}$ with a probabilistic measure P on it, so that $\forall k \in \{1, \dots, n\} : P(x_k) = \frac{1}{n}$. If for any $x_0 \in (X, \rho)$ we define a random variable $\xi_{x_0}(x) := \rho(x, x_0), x \in \Omega$, then Fermat–Torricelli set is the set of those $x_0 \in (X, \rho)$, for which random variable ξ_{x_0} has the least mathematical expectation.

Theorem 3. *Let A be the Fermat–Torricelli set for a collection $\{x_1, \dots, x_n\}$, and B be the Fermat–Torricelli set for a collection $\{y_1, \dots, y_k\}$ in Euclidean metric space (\mathbb{R}^m, ρ) with standard metric. Assume that all points $x_1, \dots, x_n, y_1, \dots, y_k$ are mutually distinct. Then if $A \cap B \neq \emptyset$, then $A \cap B$ is the Fermat–Torricelli set for $\{x_1, \dots, x_n, y_1, \dots, y_k\}$.*

Since now we will name Euclidean metric space (\mathbb{R}^2, ρ) with standard metric merely Euclidean plane.

Corollary 4. *If mutually distinct points x_1, x_2, x_3, x_4 in the Euclidean plane are vertices of a convex quadrilateral, then the point of intersection of its diagonals is a unique Fermat–Torricelli point for x_1, x_2, x_3, x_4 .*

Corollary 5. *Let x_1, x_2, x_3, x_4 be mutually distinct points in the Euclidean plane laying on the same line in the given order. Then Fermat–Torricelli set of these points is the following set*

$$A = \{\alpha x_2 + (1 - \alpha)x_3 \mid \alpha \in [0; 1]\}.$$

Corollary 6. *Let x_1, x_2, x_3 be the vertices of some triangle in the Euclidean plane, and x_4 be some other point which lays on the side of triangle between x_2 and x_3 . Then x_4 is a unique Fermat–Torricelli point of x_1, x_2, x_3, x_4 .*

Corollary 7. Let $A_1A_2\dots A_n$ be a regular polygon with an even number of vertices. Then its center of gravity is a unique Fermat–Torricelli point of its vertices.

Theorem 8. Let $x_1, \dots, x_n, n \geq 3$ be distinct points in the Euclidean plane. Then the following statements hold.

- 1) If x_1, \dots, x_n lay on the same line in the given order and n is any even number, then the set

$$A := \{x \in X \mid x = \alpha x_{\frac{n}{2}} + (1 - \alpha)x_{\frac{n}{2}+1}, \alpha \in [0; 1]\}$$

is a Fermat–Torricelli set for them.

- 2) If x_1, \dots, x_n lay on the same line in the given order and n is an odd number, then the point $x_{\frac{n-1}{2}+1}$ is a Fermat–Torricelli set for them.
- 3) If x_1, \dots, x_n do not lay on the same line, then their Fermat–Torricelli point is unique.

Lemma 9. There is a unique point inside of triangle from which every side of triangle is visible under angle 120° if and only if every angle of this triangle is less than 120° .

Different sources name this point in different ways: *Fermat point*, *Torricelli point*, and even *Steiner point* [8]. We will define it as *Steiner point for respective triangle*.

Theorem 10 ([8]). Let x_1, x_2, x_3 be vertices of triangle in Euclidean plane every angle of which is less than 120° . Then the Steiner point for this triangle is a unique Fermat–Torricelli point for x_1, x_2, x_3 .

Theorem 11 ([8]). Let x_1, x_2, x_3 be vertices of triangle in Euclidean plane one of whose angles is not less than 120° . Then the vertice, whose angle of triangle is not less than 120° , is a unique Fermat–Torricelli point of its vertices.

REFERENCES

- [1] L. Dalla, “A note on the Fermat–Torricelli point of a d -simplex,” *J. Geom.*, vol. 70, no. 1-2, pp. 38-43, 2001.
- [2] A. N. Zachos, “An analytical solution of the weighted Fermat–Torricelli problem on the unit sphere,” 2014.
- [3] A. Zachos and G. Zouzoulas, “The weighted Fermat–Torricelli problem for tetrahedra and an “inverse” problem,” *Journal of Mathematical Analysis and Applications*, vol. 353, pp. 114-120, May 2009.
- [4] F. Plastria, “Four-point Fermat location problems revisited. New proofs and extensions of old results,” *IMA Journal of Management Mathematics*, vol. 17, pp. 387-396, Oct. 2006.
- [5] S. S. Dragomir, D. Comănescu, and E. Kikianty, “Torricellian points in normed linear spaces,” *J. Inequal. Appl.*, pp. 2013:258, 15, 2013.
- [6] S. D. Nguyen, “Constrained Fermat–Torricelli–Weber Problem in real Hilbert Spaces,” 2018.
- [7] J. Sekino, “ n -ellipses and the minimum distance sum problem,” *Amer. Math. Monthly*, vol. 106, no. 3, pp. 193-202, 1999.
- [8] V. Boltyanski, H. Martini, and V. Soltan, *Geometric Methods and Optimization Problems*. Springer US, 1999.

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