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International scientific online conference

«Algebraic and geometric methods of analysis»

May 25-28, 2026.

The purpose of this conference is to bring together researchers in geometry, topology, algebra, analysis and dynamical systems and to provide for them a forum to present their recent work to colleagues from different nationalities. This way we aim to stimulate discussion about the latest findings in geometrical and topological methods in analysis and to increase international collaboration.

The conference continues the traditional annual conference «Geometry in Odesa» holding from 2004, and hosted by Odesa National University of Technology (Odesa National Academy of Food Technologies till 2021). From 2017 the conference was renamed to «Algebraic and geometric methods of analysis» (AGMA).

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- Algebraic methods in geometry
- Differential geometry in the large
- Geometry and topology of differentiable manifolds
- General and algebraic topology
- Dynamical systems and their applications
- Geometric and topological methods in natural sciences
- Geometric problems in mathematical analysis

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Homotopy type of stabilizers of Morse-Bott functions on surfaces

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Let M be a smooth surface, and P be either a real line $\mathbb{R}$ or a circle S^1. The group of diffeomorphisms $\mathcal{D}(M)$ of M acts on the space of P-valued functions on M by the following rule:

$$\zeta : C^\infty(M,P) \times \mathcal{D}(M) \to C^\infty(M,P), \qquad \zeta(f,h) = f \circ h.$$

For $f \in C^\infty(M,P)$ denote by $\mathcal{S}(f)$ the stabilizer of f with respect to the action ζ, i.e., $\mathcal{S}(f) = \{h \in \mathcal{D}(M) \mid f \circ h = f\}$. Endow strong Whitney topology on $\mathcal{D}(M)$; this topology induces some topology on $\mathcal{S}(f)$. Denote by $\mathcal{S}_{\mathrm{id}}(f)$ a path component of $\mathcal{S}(f)$ containing id_M.

For a large class of smooth functions with isolated singularities wich contains the class of Morse functions locally constant on the boundary, the homotopy type of $\mathcal{S}_{\mathrm{id}}(f)$ is known — $\mathcal{S}_{\mathrm{id}}(f)$ is either contractible or has the homotopy type of a circle S^1, see Theorem 1.3 [1]. The follolwing theorem generalizes this result for the class of Morse-Bott functions.

Theorem 1 (Theorem 1.2 [2]). *Let M be smooth, connected, compact and orientable surface and let $f : M \to P$ be a Morse-Bott function on M locally constant on ∂M. Then $\mathcal{S}_{\mathrm{id}}(f)$ is either contractible or has the homotopy type of S^1. In particular, $\mathcal{S}_{\mathrm{id}}(f)$ is contractible, if f has a saddle, and $\mathcal{S}_{\mathrm{id}}(f)$ has the homotopy type of S^1 otherwise.*

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HOL, CFD and AI as integrated methods of analysis

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Since the mid-1990s, algebraic and geometric methods of analysis have been implemented in digital simulations with exponentially increasing efficacy in STEM applications. The outcomes are so accurate that many decisions about engineering questions like, e.g., selecting a three-blade marine impeller instead of a Rushton turbine as mixer in the batch crystallization of the citric acid monohydrate (CAM), could rely solely on Artificial Intelligence (AI) and Computer Fluid Dynamics (CFD). Nevertheless, as a pivotal experience by the Department of Chemical Engineering at the University "La Sapienza" of Rome (DICMA) plainly showed [1], Hands-On Laboratory (HOL) enriches the comprehension of a process [2] revealing crucial issues about safety and sustainability [3]. La Sapienza's traditional DICMALAB confirmed that the three-blade marine impeller gave an optimal crystal size distribution (CSD) thanks to the suspension state that the axial flow provides for the dispersed phase of CAM particles [4]. Vice versa, the Rushton turbined supplied bad CSD of CAM and posed serious hazards in terms of system stability, power consumption, and noise pollution. We infer that, whenever it is possible, HOL should integrate AI & CFD as method of analysis in the investigation of STEM problems.

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Local behavior of ring Q-homeomorphisms with respect to the p-modulus under exponential integrability of the function Q

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Let a family of paths Γ be given in the complex plane $\mathbb{C}$. A Borel function $\rho : \mathbb{C} \to [0, \infty]$ is called *admissible* for Γ, denoted by $\rho \in adm\,\Gamma$, if

$$\int\limits_{\gamma} \rho(z)|dz| \geqslant 1$$

for every (locally rectifiable) path $\gamma \in \Gamma$.

Let $p > 1$. Recall that the *p-modulus* of the family of paths Γ is defined by

$$\mathrm{M}_p(\Gamma) = \inf_{\rho \in adm\,\Gamma} \int\limits_{\mathbb{C}} \rho^p(z)\, dxdy.$$

For sets E_1, E_2, and E in $\mathbb{C}$, denote by $\Delta(E_1, E_2; E)$ the family of all paths $\gamma : [a, b] \to \mathbb{C}$ that join E_1 and E_2 in E, that is, $\gamma(a) \in E_1$, $\gamma(b) \in E_2$, and $\gamma(t) \in E$ for all $t \in (a, b)$.

Let G be a domain in the complex plane $\mathbb{C}$, let $z_0 \in G$, and let $Q : G \to [0, \infty]$ be a measurable function. A homeomorphism $f : G \to \mathbb{C}$ is called a *ring Q-homeomorphism with respect to the p-modulus at the point z_0* if

$$\mathrm{M}_p(\Delta(fC_1, fC_2; f\mathbb{A})) \leqslant \int\limits_{\mathbb{A}} Q(z) \cdot \eta^p(|z - z_0|)\, dxdy$$

for every ring $\mathbb{A} = \mathbb{A}(z_0, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$, where $0 < r_1 < r_2 < \mathrm{dist}(z_0, \partial G)$, for the circles $C_1 = \{z \in \mathbb{C} : |z - z_0| = r_1\}$, $C_2 = \{z \in \mathbb{C} : |z - z_0| = r_2\}$, and for every measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that $\int\limits_{r_1}^{r_2} \eta(r)\, dr = 1$.

Further, we use the following notation:

$$B_r = \{z \in \mathbb{C} : |z| \leqslant r\}, \quad \mathbb{B} = \{z \in \mathbb{C} : |z| < 1\}.$$

In the following theorem, an upper estimate for the area of the image of a disk under ring Q-homeomorphisms with respect to the p-modulus is obtained.

Theorem 1. *Let $f : \mathbb{B} \to \mathbb{C}$ be a ring Q-homeomorphism with respect to the p-modulus at the point $z_0 = 0$ and let $1 < p < 2$. Suppose that for some numbers $\lambda > 0$ and $r_0 \in (0, 1)$ the following condition is fulfilled:*

$$Q_0 = \int\limits_{B_{r_0}} \exp\left(\lambda Q(z)\right) dxdy < \infty.$$

Then for all $r \in (0, r_0/2)$ the estimate

$$|fB_r| \leqslant c_0\, |B_r| \left(\log \frac{Q_0}{4|B_r|}\right)^{\frac{2}{2-p}},$$

holds, where $c_0 = \left(\frac{2}{\lambda}\right)^{\frac{2}{2-p}} \left(\frac{p-1}{2-p}\right)^{\frac{2(p-1)}{2-p}}$.

Below we present a theorem on the power-logarithmic asymptotics in terms of lower limits for ring Q-homeomorphisms with respect to the p-modulus at the point $z_0 = 0$ for $1 < p < 2$.

Theorem 2. *Let $f : \mathbb{B} \to \mathbb{C}$ be a ring Q-homeomorphism with respect to the p-modulus at the point $z_0 = 0$, let $1 < p < 2$, and let $f(0) = 0$. Assume that for some numbers $\lambda > 0$ and $r_0 \in (0, 1)$ the condition*

$$Q_0 = \int\limits_{B_{r_0}} \exp\left(\lambda Q(z)\right) dxdy < \infty$$

holds. Then

$$\liminf_{z \to 0} \frac{|f(z)|}{|z| \left(\log \frac{1}{|z|}\right)^{\frac{1}{2-p}}} \leqslant k_0,$$

where $k_0 = \left(\frac{4}{\lambda}\right)^{\frac{1}{2-p}} \left(\frac{p-1}{2-p}\right)^{\frac{p-1}{2-p}}$.

Research supported by the National Research Foundation of Ukraine, Project number 2025.07/0014, Project name: "Modern problems of Mathematical Analysis and Geometric Function Theory".

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From Stars to Double-Stars via Ultrametrics

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By labeled tree $T(l)$ we mean a tree T equipped with a labeling $l\colon V(T) \to \mathbb{R}^+$, where $V(T)$ is the vertex set of T. Let $T(l)$ be a labeled tree. As in [1] we define a mapping $d_l\colon V(T) \times V(T) \to \mathbb{R}^+$ by

$$d_l(u,v) = \begin{cases} 0, & \text{if } u = v,\\ \max\limits_{w \in V(P)} l(w), & \text{otherwise,} \end{cases}$$

where P is the path joining u and v in T. Let (Y,ρ) be an ultrametric space. We say that (Y,ρ) is generated by $l\colon V(T) \to \mathbb{R}^+$ if the equality $(Y,\rho) = (V(T), d_l)$ holds.

The concept of ultrametric spaces generated by non-negative vertex labelings on both finite and infinite trees was introduced in [1] and investigated in [3, 4, 7, 9]. Recently, in [5], the centers of distances of ultrametric spaces generated by labeled trees were described. The ultrametric spaces generated by vertex labeled star graphs are studied in [2, 8, 9, 6]. Here we would like to present a solution of the following problem.

Problem 1. Describe the structure of trees T which satisfy the following condition: If d is an arbitrary ultrametric generated by vertex labeling of T, then the ultrametc space $(V(T), d)$ is isometric to an ultrametric space generated by vertex labeling of some star graph.

The following theorem gives a solution of Problem 1.

Theorem 2. *Let T be a tree, $U(T)$ be the set of all ultrametric spaces generated by vertex labelings of T, and let* **US** *be the class of all ultrametric spaces generated by labelings of star graphs. Then the following statements are equivalent:*

(i) The inclusion $U(T) \subseteq \mathbf{US}$ holds.
(ii) The length of the longest path in T does not exceed three.

The above theorem implies the following corollary.

Corollary 3. *Let T be a tree. Then the inclusion $U(T) \subseteq \mathbf{US}$ holds if and only if T is isomorphic either to a star-graph or to a double-star graph.*

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The Structure of D-homoderivations in Lie Algebra

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Lie algebras and their derivations are central to many areas of mathematics and theoretical physics [1]. The study of derivations, defined as linear maps that satisfy the Leibniz rule with respect to the Lie bracket, has expanded to include generalized derivations, quasi-derivations, centroids, and central derivations [2]. These generalizations offer a deeper understanding of the symmetry and automorphism properties of Lie algebras. Recent research has advanced the understanding of derivation-like structures under various algebraic constructions. Benkovič and Eremita [2] investigated conditions under which the decomposition $QDer(L) = Der(L) \oplus C(L)$ holds for quasi-derivations in current Lie algebras, particularly in the tensor product $L \otimes A$.

The study of derivation was initiated during the 1950s and 1960s. Strictly, derivations of rings got a tremendous development in 1957. Based on the fundamental definition of derivation $\zeta\colon \mathcal{G} \to \mathcal{G}$ satisfies $\zeta(xy) = \zeta(x)y + x\zeta(y)$ where $x, y \in \mathcal{G}$. When the year 2000 came, a classical definition concerning of homoderivation was delivered, where an additive mapping ζ is a homoderivation concerning a ring $\mathcal{G}$. More precisely, $\zeta\colon \mathcal{G} \to \mathcal{G}$ satisfies $\zeta(xy) = \zeta(x)\zeta(y) + \zeta(x)y + x\zeta(y)$ where x and y in $\mathcal{G}$. Lastly, in 2025, Mehsin Jabel Atteya via [4] contributed some promising results concerning the behavior of homogeneralized (σ, τ)-derivations and homogeneous (μ, ξ)-homoderivations on associative rings.

This paper presents explicit structural results concerning D-homoderivations in Lie algebras over arbitrary fields. It is established that the set of D-homoderivations forms a Lie algebra, which decomposes as the sum of homoderivations and centroids, intersecting precisely at the space of central homoderivations.

Definition 1. A linear mapping $\zeta\colon L \to L$ for any Lie algebra L is a D-homoderivations Lie algebra for which there exists a homoderivation D such that

$$\zeta([x, y]) = [\zeta(x), D(y)] + [\zeta(x), y] + [x, D(y)]$$

where $x, y \in L$. We denote by $Der_{HD}(L)$ for a D-homoderivations Lie algebra.

Theorem 2. *The $Der_{HD}(L)$ is a Lie subalgebra of $gl(L)$.*

Due to $Der_{HD}(L)$ forms a Lie algebra, it is clear that $Der(L)$ is a Lie subalgebra of $Der_{HD}(L)$. We now establish a decomposition of D-homoderivations into ordinary derivations and centroids.

Theorem 3. *Let L be a Lie algebra. Then, $Der_{HD}(L) = Der_{HD}(L) + C(L)$.*

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Soliton Analysis of some PDEs by Different Algebraic Methods.

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descriptions of nonlinear phenomena and ever-changing dynamics can be realized through mathematical modeling, which incorporates different differential equations. Because of these points, this study has examined the soliton propagation of the Fokas, Gilson Pickering and BLMP equation. The new soliton solutions are extracted, such as solitary, dark, bright, singular, periodic, and plane wave solutions, that have not been found in the literature before with the application of the extended simple equation method (ESEM) and modified F-expansion method (MFEM). The uniqueness and differentiation of our solutions lie in the specific constraint conditions and parameter ranges considered in our study. The investigation shows that the approach used to acquire complete and distinctive solutions rapidly is successful. Based on the proposed schemes offered herein, these results can be measured as directions for potential future research through systematized and appropriate engineering and applied science methods. In future, i am planning to write a book on these results so that this research will be helpful in true sense.

Singularities of real nodal curves

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It is a joint work with Igor Burban. For details see [2].

Let $(X, \mathcal{A})$ be a reduced non-commutative curve [3], that is X is an algebraic curve over a field $\Bbbk$ and $\mathcal{A}$ is a sheaf of $\mathcal{O}_X$-algebras coherent and torsion free as $\mathcal{O}_X$-module and without nil-ideals. Without loss of generality, we can suppose that this non-commutative curve is *central*, that is $\mathcal{O}_x$ coincides with the center of $\mathcal{A}_x$ for every point $x \in X$. In "usual" (commutative) case it means that $\mathcal{A} = \mathcal{O}_X$.

We denote by $\mathcal{K}$ the sheaf of rational functions on X, set $\mathcal{K}\mathcal{A} = \mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{A}$. Note that for every $x \in X$ the localization $\mathcal{A}_x$ is an $\mathcal{O}_x$-*order* in $\mathcal{K}\mathcal{A}$. We say that a point $x \in X$ is *singular for* $\mathcal{A}$ if $\mathcal{A}_x$ is not a maximal order. We call the completion $\hat{\mathcal{A}}_x$ of this localization a *singularity* of the non-commutative curve $(X, \mathcal{A})$. This singularity is called *nodal* if

(1) $\mathrm{End}_{\mathcal{A}_x} \mathcal{J}_x = \mathcal{H}_x$ is hereditary, where $\mathcal{J}_x = \mathrm{rad}\mathcal{A}_x$,
(2) $\mathrm{rad}\mathcal{H}_x = \mathcal{J}_x$,
(3) $\ell_{\mathcal{A}_x}(\mathcal{H}_x \otimes_{\mathcal{A}_x} U) \leq 2$ for every simple $\mathcal{A}_x$-module U.

(Note that these conditions hold for $\mathcal{A}_x$ if and only if they hold for $\hat{\mathcal{A}}_x$).

It is known [1] that up to Morita equivalence the non-commutative curve $(X, \mathcal{A})$ is defined by the $\mathcal{K}$-algebra $\mathcal{K}\mathcal{A}$ and the rings $\mathcal{A}_x$ for the points x singular for $\mathcal{A}$.

If the basic field $\mathbb{k}$ is algebraically closed, the possible singularities were classified by Voloshyn [4]. We describe a general procedure that allows to construct nodal singularities in general case and classify them if the basic field is the field of real numbers.

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On explicit Reeb spaces of real analytic functions which are not homeomorphic to any finite graph

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Since the last century, *Reeb spaces* of (continuous real-valued) functions have been fundamental and important tools. They are in nice cases graphs (*Reeb graphs*: [10]). In theory of Morse functions and naturally generalized functions and applications to differential topology and geometric topology of manifolds, they have been also fundamental and important topological and combinatorial objects.

Definition 1. For a continuous real-valued function $c : X \to \mathbb{R}$ on a topological space X, by the following equivalence relation $\sim_c$ on X, we have the quotient space $R_c := X/\sim_c$ and this is the *Reeb space* of c: $x_1 \sim_c x_2$ if and only if they are in a same connected component of a same level set $c^{-1}(y)$. We have the quotient map $q_c : X \to R_c$ and the unique continuous function $\bar{c} : R_c \to \mathbb{R}$ with $c = \bar{c} \circ q_c$.

Theorem 2 ([11])**.** *If X is a smooth closed and connected manifold and the critical value set of c is finite, then R_c has the structure of a finite and connected graph whose vertex set consists of all points v with ${q_c}^{-1}(v)$ containing some critical points of c* ([11, Theorem 3.1]).

For more explicit cases, see [2] for example. We call a graph obtained in the rule above, the *Reeb graph* of a smooth function $c : X \to \mathbb{R}$, where X is with no boundary. The speaker has been interested in reconstruction of nice smooth or real algebraic functions whose Reeb graphs are isomorphic to given finite graphs and has contributed to this. We review related history and present our main problem (Problem 3).

- In [13], Sharko first considers reconstruction of a nice smooth function on a closed surface whose Reeb graph is isomorphic to a given finite graph of a certain natural class. Critical points of the function are isolated and around them, the function is represented by certain elementary polynomials or Morse. In [8], Masumoto and Saeki have extended this to arbitrary finite graphs and the critical sets of the functions may not be isolated.

- In [9], Michalak has considered reconstruction of Morse functions whose level sets with no critical point are disjoint unions of spheres and he has classified Reeb (di)graphs of Morse functions on each closed surface.
- The author has launched studies on reconstruction of nice smooth functions with given Reeb graphs, and as additional data, given level sets. Related studies on smooth functions on closed manifolds are [3, 6], for example. For the non-compact case, see [4] for example.
- The author has launched studies on reconstruction of explicit real algebraic functions with given Reeb graphs, in [5]. Here, a *real algebraic* map is the canonical embedding of a union of connected components of the zero set of a real polynomial map, or its composition with the canonical projection $\pi_{k_1,k_2} : \mathbb{R}^{k_1} \to \mathbb{R}^{k_2}$ with $k_1 > k_2 \geq 1$ and $\pi_{k_1,k_2}(x_1, x_2) := x_1$. It is a fundamental and important fact that in the real algebraic (analytic) category, we cannot use tools or methods such as bump functions, the partition of the unity, and so on. In smooth (differentiable) situations, we locally construct nice smooth functions and glue them together to have desired functions and we cannot apply this in algebraic (analytic) cases.

Problem 3. For (Reeb) spaces which are not finite graphs, can we reconstruct nice smooth functions?

Some examples of smooth functions which are not real analytic have been presented in [11, 12] for example. Different from such cases, we consider real analytic maps or smooth functions which are real analytic on dense subsets and each component of which is an elementary function.

Theorem 4 ([7]). *There exists an infinite graph G_1, for any integer $m \geq 2$, there exists an m-dimensional smooth submanifold $X_m \subset \mathbb{R}^{m+1}$ with no boundary which is represented as the zero set ${e_m}^{-1}(0)$ of some real analytic and elementary function $e_m : \mathbb{R}^{m+1} \to \mathbb{R}$, and the following hold.*

(1) The Reeb space $R_{\pi_{m+1,1}|_{X_m}}$ is regarded as the Reeb graph and isomorphic to G_1.

(2) The quotient map $q_{\pi_{m+1,1}|_{X_m}}$ is proper and the preimage of each point having no critical point of the function $\pi_{m+1,1}|_{X_m}$ is diffeomorphic to S^{m-1}.

According to [1], the Reeb space R_c of a smooth function $c : X \to \mathbb{R}$ on a closed and connected manifold is a compact, connected and locally connected metrizable space (*Peano continuum*).

Theorem 5 ([7]). *There exists a Peano continuum G_2 which is not homeomorphic to any graph, for any pair (m_1, m_2) of integers $m_1, m_2 \geq 2$, there exists an $(m_1 + m_2)$-dimensional smooth compact submanifold $X_{m_1,m_2} \subset \mathbb{R}^{m_1+m_2+2}$ with no boundary represented as the zero set ${e_{m_1,m_2}}^{-1}(0)$ of some smooth map $e_{m_1,m_2} : \mathbb{R}^{m_1+m_2+2} \to \mathbb{R}^2$ which is real analytic and each component of which is an elementary function outside some subset $Z_{m_1,m_2} \subset \mathbb{R}^{m_1+m_2+2}$ of Lebesgue measure 0, and the following hold.*

(3) For some point p_{G_2}, $G_2 - \{p_{G_2}\}$ is homeomorphic to a graph, the preimage ${q_{\pi_{m_1+m_2+2,1}|_{X_{m_1,m_2}}}}^{-1}(p_{G_2})$ is $\{0\} \subset \mathbb{R}^{m_1+m_2+2}$, and $R_{\pi_{m_1+m_2+2,1}|_{X_{m_1,m_2}-\{0\}}}$ is regarded as the Reeb graph of $\pi_{m_1+m_2+2,1}|_{X_{m_1,m_2}-\{0\}}$.

(4) The Reeb space $R_{\pi_{m_1+m_2+2,1}|_{X_{m_1,m_2}}}$ is homeomorphic to G_2.

The speaker is a researcher at Osaka Central Advanced Mathematical Institute, or a member of OCAMI researchers, and this is supported by MEXT Promotion of Distinctive Joint Research Center Program JPMXP0723833165. He thanks this opportunity, where he is not employed there.

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Algebraic and Geometric Aspects of Non-Classical Knots

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Classical knot theory is built on the principle that crossings in a diagram can be freely manipulated via Reidemeister moves. In recent years, several extensions of this framework have emerged by modifying the behavior of crossings, leading to new classes of knot-like objects. In this talk, we survey a range of such non-classical knot theories, including *pseudo knots* [6], *singular knots* [4] and *bonded knots* [2], emphasizing their connections with braid theory and algebraic structures [1, 5].

We then introduce the notion of *stuck knots* [1], where certain crossings are rigid and cannot be altered by Reidemeister moves. This rigidity leads to new phenomena, including modified equivalence relations, novel invariants and the concept of unsticking distance. We discuss how classical tools, such as skein relations and HOMFLYPT-type invariants, extend to these settings, highlighting the robustness of algebraic constructions across different crossing behaviors.

Finally, we outline a program toward a theory of stuck braids, aiming to establish analogues of Alexander and Markov theorems and to develop associated algebraic frameworks. This perspective suggests a unifying approach to studying knot theory through the lens of crossing behavior, bridging geometry, algebra and emerging computational directions.

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Supersymmetry, Supergravity and Non–Perturbative Dynamics of Gauge Theories

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Supersymmetry extends the Poincaré algebra [1] by introducing fermionic generators Q_α and $\bar{Q}_{\dot\alpha}$ satisfying

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu, \qquad \{Q_\alpha, Q_\beta\} = 0, \tag{1}$$

which enforce a correspondence between bosonic and fermionic degrees of freedom. This algebraic structure leads to cancellations in quantum corrections and constrains the form of effective actions.

In $\mathcal{N} = 1$ supersymmetric theories, the dynamics is encoded in the superspace action

$$S = \int d^4x\, d^4\theta\, K(\Phi_i, \Phi_i^\dagger) + \left[\int d^4x\, d^2\theta\, W(\Phi_i) + \text{h.c.}\right], \tag{2}$$

where K is the Kähler potential and W is the holomorphic superpotential. The scalar potential takes the universal form

$$V = \sum_i \left|\frac{\partial W}{\partial \phi_i}\right|^2 + \frac{g^2}{2}\sum_a \left(\phi^\dagger T^a \phi\right)^2. \tag{3}$$

A remarkable realization of exact non–perturbative dynamics occurs in $\mathcal{N} = 2$ supersymmetric Yang–Mills theory, where the low-energy effective action is determined by a single holomorphic prepotential $\mathcal{F}(a)$. The physical variables are given by period integrals

$$a = \oint_\alpha \lambda_{\text{SW}}, \qquad a_D = \oint_\beta \lambda_{\text{SW}}, \tag{4}$$

with the Seiberg–Witten differential [2, 3]

$$\lambda_{\text{SW}} = \frac{x\,dx}{y}, \tag{5}$$

defined on the elliptic curve

$$y^2 = (x - e_1)(x - e_2)(x - e_3). \tag{6}$$

The BPS spectrum is exactly determined by the central charge

$$Z = n_e a + n_m a_D, \qquad M_{\text{BPS}} = |Z|, \tag{7}$$

and electric–magnetic duality acts as an $SL(2, \mathbb{Z})$ transformation

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix}, \qquad ps - qr = 1. \tag{8}$$

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Coupling to gravity promotes global supersymmetry to local supersymmetry, yielding $\mathcal{N} = 1$ supergravity with scalar potential

$$V = e^{K/M_{\mathrm{Pl}}^2} \left(K^{i\bar{j}} D_i W D_{\bar{j}} \bar{W} - \frac{3}{M_{\mathrm{Pl}}^2} |W|^2 \right), \tag{9}$$

where

$$D_i W = \partial_i W + \frac{1}{M_{\mathrm{Pl}}^2} (\partial_i K) W. \tag{10}$$

In string compactifications [4], moduli stabilisation can be achieved via the KKLT mechanism with superpotential

$$W = W_0 + A e^{-aT}, \tag{11}$$

leading to a scalar potential whose structure depends sensitively on higher-order corrections. In particular, α'^3 corrections modify the Kähler potential as

$$K = -2 \ln \left(\mathcal{V} + \frac{\hat{\xi}}{2} \right), \tag{12}$$

breaking the no-scale identity

$$K^{T\bar{T}} (\partial_T K)(\partial_{\bar{T}} K) = 3, \tag{13}$$

and generating distinct regimes of the potential, including metastable de Sitter vacua and runaway solutions.

This framework reveals a deep interplay between algebra, geometry, and quantum dynamics, where gauge theory observables are encoded in geometric data and vacuum structure is governed by holomorphic functions. The results provide a unified perspective on non–perturbative field theory, supergravity, and string compactifications, and shed light on fundamental questions concerning vacuum stability and the structure of the landscape.

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Problem with integral condition for homogeneous system of partial differential equations of fourth order in Sobolev space

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In the strip $\Omega(T):\{(t,x)\in\mathbb{R}^{n+1}: t\in(0,T), x\in\mathbb{R}^n\}$ we consider nonlocal problem with integral conditions

$$L\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x}\right)\overrightarrow{U}(t,x)=\frac{\partial^4\overrightarrow{U}(t,x)}{\partial t^4}+\sum_{i=1}^{4}A_i\frac{\partial^4\overrightarrow{U}(t,x)}{\partial t^{4-i}\partial x^i}=\overrightarrow{0}, \tag{1}$$

$$\int\limits_{T_1}^{T_2} t^k\overrightarrow{U}(t,x)dt=\overrightarrow{\varphi}_j(x), \quad k=\{0,1,2,3\}, \tag{2}$$

where $\overrightarrow{U}(t,x)=col\,(U^1(t,x),...,U^4(t,x))$, $\overrightarrow{\varphi}_j(x)=col\left(\varphi_j^1,...,\varphi_j^4\right)$, $j=\{1,2,3,4\}$, A_i, are squert matrix 4×4,

$$\int\limits_0^T t^k\overrightarrow{U}(t,x)dt=col\left(\int\limits_0^T t^kU_1(t,x)dt,...,\int\limits_0^T t^kU_4(t,x)dt\right).$$

Main determinand of the system (1) in the form $\Delta(\xi)$. Let $H_\alpha, \alpha>0$ be a Sobolew space with the finite norm [1].

$$\|\varphi(x);H_\alpha\|=\sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty}(1+|\xi|)^{2\alpha}\,|\widetilde{\varphi}(\xi)|^2d\xi}<\infty$$

where $\widetilde{\varphi}(\xi)$ is a Fourier transformation of the funktion $\varphi(x)$. $\overline{H}_\alpha$ is a vector space of the funktion $\varphi(x)$, with norm

$$\|\varphi(x),\overline{H}_\alpha\|=\max\|\varphi^j(x);H_\alpha\|$$

Theorem. Let conditions occur, let $\Delta(\xi)\neq0$ for everyone $\xi\in\mathbb{R}\setminus\{0\}$. If $\overrightarrow{\varphi}_j\in\overline{H}^4_{\alpha_1}$, $\alpha_1\geqslant\alpha_2+4\,(C_4^4+1)$, $\alpha_2>1$, $j=\{1,...,4\}$, then the space $C^4(0,T),\overline{H}^4_\alpha)$ exists and unique solution $\overrightarrow{U}(t,x)$ of the problem (1)-(2), which still is depending on vector-function $\overrightarrow{\varphi}_j(x)$, $j=\{1,...,4\}$. Solution is the represented in the form

$$\overrightarrow{U}(t,x)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}e^{ix\xi}\sum_{j,q=1}^{16}\frac{\Delta_{j,q}(\xi)}{\Delta(\xi)}e^{\lambda_q\xi}\overrightarrow{h}_q\psi_j(\xi)d\xi, \tag{3}$$

where $\Delta_{j,q}(\xi)$, $j,q=\{1,\ldots,4\}$, $\xi\neq 0$, – algebraic complement the standing element j this poem and q for this column of indicator $\Delta(\xi)$,

$$\mathrm{col}(\psi_1(\xi),\ldots,\psi_4(\xi))=\mathrm{col}(\tilde{\varphi}_1^1(\xi),\ldots,\tilde{\varphi}_1^4(\xi);\ldots;\tilde{\varphi}_4^1(\xi),\ldots,\tilde{\varphi}_4^4(\xi)),$$

i $\tilde{\varphi}_j^q(\xi)$ is a Fourier transformation of the funktion $\varphi_j^q(x)$, $j=\{1,...,4\}$, $q=\{1,...,4\}$. This result continues the research of work [2-9].

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Numerical moduli in the geometry of [special] multi-flags

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A number of key issues concerning distributions generating 1-flags (most often called Goursat flags) has been settled over the past 30 years.
Presently similar questions are being discussed as regards distributions generating *multi*-flags. (More precisely, only so-called *special* multi-flags, to avoid functional moduli in local classifications.) In particular, special 2-flags of small lengths are a natural ground for the search of generalizations of theorems established earlier for Goursat structures. This includes the search for the first appearing modulus (or moduli) in the classification up to local diffeomorphisms of special 2-flags. (For Goursat flags the first modulus of the local classification appears in length 8.)
It has been known in this respect that up to length 4 that classification is finite ([3], [2]), and that in length 7 at least one numerical modulus exists, as produced on pp. 39 – 41 in [3]. (That result from the year 2010 went unnoticed in [1].)
In the last fully classified length 4 possible are precisely 34 local geometries (local models) of special 2-flags, firstly described in [3].

In the year 2024 we have demonstrated (cf. [4]) that in the length 5 single numerical moduli show up in exactly three out of altogether 41 singularity classes of special 2-flags existing in that length.

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A note on demi Dunford-Pettis operators

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Let X and Y be Banach spaces and X', Y' be norm dual of X, Y, respectively. A linear bounded operator $T : X \to Y$ is called a Dunford-Pettis operator if $x_n \to 0$ weakly in X implies $\| Tx_n \| \to 0$ in Y. Many authors are interested in this subject. Adjoint operator of $T : X \to Y$ is defined by $T' : Y' \to X'$, $(T'f)(x) = f(Tx)$ for every $f \in Y', x \in X$. An operator $T : X \to X$ is said to be demi Dunford-Pettis operator if, for every sequence (x_n in X such that $x_n \to 0$ weakly in X with $\|x_n - Tx_n\| \to 0$,then $\|x_n\| \to 0$ in X.Every Dunford-Pettis operator is a demi Dunford-Pettis operator. An operator $T : X \to Y$ is called a weakly Dunford-Pettis operator if, for every sequence (x_n) in X with $x_n \to 0$ weakly in X and (f_n) in Y' with $f_n \to 0$ weakly in Y' , then $f_n(Tx_n) \to 0$ as $n \to \infty$. We define weakly demi Dunford-Pettis operator that it is different from defined by J. Riberio and F. Santos. We show that the results obtained by J. Riberio and F. Santos are also true by using our definition.Also, we investigate the results on Banach lattices.

Theorem 1 ([2]). *Every weakly Dunford-Pettis operator is a weakly demi Dunford-Pettis operator.*

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Evolutionary inner radii estimates for systems of domains with given parameters

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This work investigates the problem of estimating functionals defined on systems of mutually non-overlapping domains in the complex plane. In particular, we obtain upper bounds for products of the inner radii of n such domains with respect to systems of fixed points. The main result establishes the dependence of these upper bounds on a real parameter γ for the entire range of its values $\gamma \in (0, n]$.

Let $r(B,a)$ be the inner radius of a domain $B \subset \overline{\mathbb{C}}$ with respect to a point $a \in B$. The inner radius is a generalization of the conformal radius to the case of multiply connected domains. The inner radius of the domain B is related to the generalized Green function $g_B(z,a)$ of the domain B by the following relations:

$$g_B(z,a) = \ln \frac{1}{|z-a|} + \ln r(B,a) + o(1), \quad z \to a,$$

$$g_B(z,\infty) = \ln |z| + \ln r(B,\infty) + o(1), \quad z \to \infty.$$

This work is devoted to obtaining evolutionary-type inequalities for functionals of the following form for all values of the parameter $\gamma \in (0, n]$:

$$I_n(\gamma) = r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

$$Y_n(\gamma) = r^{\gamma}\left(B_\infty, \infty\right) \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

$$J_n(\gamma) = \left(r\left(B_0, 0\right) r\left(B_\infty, \infty\right)\right)^{\gamma} \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

$$I_n(\gamma_1, \ldots, \gamma_n) = \prod_{k=1}^{n} r^{\gamma_k}(B_k, a_k),$$

where $n \in \mathbb{N}$, $\{a_k\}_{k=1}^n$ is an arbitrary fixed system of points in the complex plane $\mathbb{C}\backslash\{0\}$; B_0, B_∞, $\{B_k\}_{k=1}^n$ is an arbitrary system of pairwise non-overlapping domains such that $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, and $a_k \in B_k \subset \overline{\mathbb{C}}$ for $k = \overline{1,n}$.

These functionals have been studied in numerous papers under various conditions (see, e.g., [1–5]), but currently, there are no methods for obtaining their exact maximums for all values of the parameter $\gamma \in (0, n]$. In particular, in [1], dynamic estimates were proven for the functionals $I_n(\gamma)$, $Y_n(\gamma)$, and $J_n(\gamma)$, expressed in terms of their initial values $I_n(0)$, $Y_n(0)$, and $J_n(0)$, respectively, and a generalization of M.A. Lavrentiev's result on the maximum product of conformal radii of two non-overlapping simply connected domains was obtained. In the present work, we obtain evolutionary estimates for the products of inner radii of pairwise non-overlapping multi-connected domains with respect to fixed points in the complex plane.

Theorem 1. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\varepsilon > 0$, $\gamma \in (0, n]$, and $\tau \in (0, \gamma)$. Then, for any fixed system of distinct points $\{a_k\}_{k=1}^n \in \mathbb{C}\backslash\{0\}$ and any collection of pairwise non-overlapping domains B_k, $k = \overline{0,n}$, such that $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{0,n}$, $a_0 = 0$, and $r\left(B_k, a_k\right) \geqslant \varepsilon$ for $k = \overline{1,n}$, the following inequality holds:*

$$I_n(\gamma) \leqslant n^{-\frac{\gamma-\tau}{2}} \varepsilon^{\tau-\gamma} I_n(\tau) \left(\prod_{k=1}^{n} |a_k|\right)^{\frac{2(\gamma-\tau)}{n}}.$$

Theorem 2. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\varepsilon > 0$, $\gamma \in (0, n]$, and $\tau \in (0, \gamma)$. Then, for any fixed system of distinct points $\{a_k\}_{k=1}^n \in \mathbb{C}$ and any collection of pairwise non-overlapping domains B_∞, B_k, such that $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, and provided that $r(B_k, a_k) \geqslant \varepsilon$ for all $k = \overline{1,n}$, the following inequality holds:*

$$Y_n(\gamma) \leqslant n^{-\frac{\gamma-\tau}{2}} \varepsilon^{\tau-\gamma} Y_n(\tau).$$

Theorem 3. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\varepsilon > 0$, $\gamma \in (0, n]$, and $\tau \in (0, \gamma)$. Then, for any fixed system of distinct points $\{a_k\}_{k=1}^{n} \in \mathbb{C}\backslash\{0\}$ and any collection of pairwise non-overlapping domains B_0, B_∞, B_k, such that $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, and $a_k \in B_k \subset \overline{\mathbb{C}}$ for $k = \overline{1,n}$, provided that $r(B_k, a_k) \geqslant \varepsilon$ for all $k = \overline{1,n}$, the following inequalities hold:*

$$J_n(\gamma) \leqslant \begin{cases} (n+1)^{-(\gamma-\tau)\frac{n+1}{n+2}}\, \varepsilon^{-\frac{2n(\gamma-\tau)}{n+2}}\, J_n(\tau) \prod\limits_{k=1}^{n} |a_k|^{\frac{2(\gamma-\tau)}{n+2}}, & \text{if } \gamma \in \left(0, \frac{n+2}{2}\right]; \\ (n+1)^{-\frac{n+1}{2}} \prod\limits_{k=1}^{n} |a_k|, & \text{if } \gamma \in \left(\frac{n+2}{2}, n\right]. \end{cases}$$

Theorem 4. *Let $n \in \mathbb{N}$, $n \geqslant 3$, and γ_k, θ_k, $k = \overline{1,n}$, be some positive real numbers such that $\gamma_k \geqslant \theta_k$ for $k = \overline{1,n}$. Then, for any fixed system of distinct points $\{a_k\}_{k=1}^{n} \in \mathbb{C}$ and any collection of domains $B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, such that $a_k \in B_k$, $k = \overline{1,n}$, $B_i \cap B_j = \emptyset$ for $i \neq j$, and provided that $r(B_k, a_k) \geqslant \varepsilon$ for $k = \overline{1,n}$ and some $\varepsilon > 0$, the following inequality holds:*

$$I_n(\gamma_1, \ldots, \gamma_n) \leqslant \varepsilon^{-\sum\limits_{k=1}^{n}(\gamma_k-\theta_k)} (n-1)^{-\frac{1}{2}\sum\limits_{k=1}^{n}(\gamma_k-\theta_k)} \prod_{i,j=1, i<j}^{n} |a_j - a_i|^{\frac{2}{n-1}(\gamma_i+\gamma_j-\theta_i-\theta_j)} \prod_{k=1}^{n} r^{\theta_k}(B_k, a_k).$$

The above Theorems allow us to obtain certain estimates of evolutionary type. These estimates can be useful, for example, when we know the exact estimates for a certain fixed configuration of points a_k and a certain set of exponents θ_k, but we need to find estimates for the same configuration of points a_k but for a different set of exponents α_k. The estimates obtained in this manner can be more precise than those obtained directly.

Acknowledgements. This study was funded by the National Research Foundation of Ukraine in the framework of the research project 2025.07/0014 on the topic "Modern problems of mathematical analysis and geometric function theory".

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All definitions and notions used below may be found in [1]. Let $S_i = S(x_0, r_i) = \{x \in \mathbb{R}^n : |x - x_0| = r_i\}$, $i = 1, 2$, $n \geqslant 2$, and let $Q : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable function satisfying the condition $Q(x) \equiv 0$ for $x \in \mathbb{R}^n \setminus D$. In what follows, $M(\cdot)$ denotes the modulus of family of paths, and $dm(x)$ is an element of the Lebesgue measure in $\mathbb{R}^n$. A mapping $f : D \to \overline{\mathbb{R}^n}$, $\overline{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty\}$, is called a *ring Q-mapping at the point* $x_0 \in \overline{D} \setminus \{\infty\}$, if the condition

$$M(f(\Gamma(S_1, S_2, D))) \leqslant \int\limits_{A \cap D} Q(x) \cdot \eta^n(|x - x_0|)\, dm(x)\,,$$

$A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$, holds for all $0 < r_1 < r_2 < d_0 := \sup\limits_{x \in D} |x - x_0|$ and all Lebesgue measurable functions $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1\,.$$

Below $h(x, y)$ denotes the chordal (spherical metric) between points $x, y \in \overline{\mathbb{R}^n}$. Let $h(E)$ be achordal diameter of the set E in $\overline{\mathbb{R}^n}$. Given $x_0 \in \overline{\mathbb{R}^n}$ and $r > 0$ we set $B_h(x_0, r) := \{x \in \overline{\mathbb{R}^n} : h(x, x_0) < r\}$. For $x_0 \in \mathbb{R}^n$ and $r_0 > 0$, as usual, $B(x_0, r_0) = \{x \in \mathbb{R}^n : |x - x_0| < r_0\}$. A family $\mathfrak{F}$ of mappings $f : D \to \overline{\mathbb{R}^n}$ is called *uniformly open at a point* $x_0 \in D$, if for every $\varepsilon_0 > 0$, $\varepsilon_0 < \operatorname{dist}(x_0, \partial D)$, there is $r_0 = r_0(x_0, \varepsilon_0) > 0$ such that $B_h(f(x_0), r_0) \subset f(B(x_0, \varepsilon_0))$ for every $f \in \mathfrak{F}$. A family $\mathfrak{F}$ of mappings $f : D \to \overline{\mathbb{R}^n}$ is called *uniformly open on* K, if for every $\varepsilon_0 > 0$ there exists $r_0 = r_0(K, \varepsilon_0) > 0$ such that $B_h(f(x_0), r_0) \subset f(B(x_0, \varepsilon_0))$ for every $f \in \mathfrak{F}$ and every $B(x_0, \varepsilon_0) \subset K$.

Given a domain D in $\mathbb{R}^n$, $n \geqslant 2$, a Lebesgue measurable function $Q : D \to [0, \infty]$, a compact set $K \subset D$ and a number $\delta > 0$ denote by $\mathfrak{F}^{\delta}_{K,Q}(D)$ a family of all ring Q-homeomorphisms $f : D \to \overline{\mathbb{R}^n}$ such that $h(f(K), \partial f(D)) = \inf\limits_{x \in f(K), y \in \partial f(D)} h(x, y) \geqslant \delta$. The following result holds.

Theorem 1. (An analogue of Koebe-Bloch theorem). *Let D be a bounded domain. Assume that, for each point $x_0 \in D$ and for every $0 < r_1 < r_2 < r_0 := \sup\limits_{x \in D} |x - x_0|$ there is a set $E_1 \subset [r_1, r_2]$ of a positive linear Lebesgue measure such that the function Q is integrable with respect to $\mathcal{H}^{n-1}$ over the spheres $S(x_0, r)$ for every $r \in E_1$. Then the family $\mathfrak{F}^{\delta}_{K,Q}(D)$ is uniformly open on K.*

Remark 2. If f is a homeomorphism, $f : D \to \mathbb{R}^n$, $n \geqslant 3$, $f \in W^{1,\varphi}_{\rm loc}$, $\int\limits_1^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-2}} dt < \infty$ and $K_O(x, f) \in L^{n-1}_{\rm loc}$, then $g := f^{-1} \in W^{1,n}_{\rm loc}$, where $K_O(x, f)$ is an outher dilatation of f at x. On

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the other hand, if $K_I(y,g) \in L^1_{\mathrm{loc}}$, then g is a ring Q_*-homeomorphism with $Q_* = K_I(y,g)$ (see Corollary 8.5, Theorem 8.6 in [1]). Thus, Theorem 1 holds for Orlicz-Sobolev classes $W^{1,\varphi}_{\mathrm{loc}}$ under above assumptions.

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Acknowledgements. This talk is supported by the National Research Foundation of Ukraine (Project "Analogues of Carathéodory and Koebe-Bloch theorems for Orlycz-Sobolev classes", Project number 2025.02/0010).

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A Riemann Boundary Value Problem in a Two-Dimensional Commutative Banach Algebra

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Let $\mathbb{B}$ be a two-dimensional commutative Banach algebra over $\mathbb{C}$ with basis $\{1, \rho\}$, where $\rho^2 = 0$, and let E be a two-dimensional real linear subspace of $\mathbb{B}$. A function Φ defined on a domain in E with values in $\mathbb{B}$ is called *monogenic* if it is Gâteaux differentiable at every point of the domain.

Let γ be a rectifiable Jordan curve in E bounding a domain D^+. We study the Riemann boundary value problem for monogenic functions: to find a monogenic function Φ in $E \setminus \gamma$ satisfying

$$\Phi^+(\tau) = G(\tau)\,\Phi^-(\tau) + g(\tau), \quad \tau \in \gamma,$$

where G, g are given $\mathbb{B}$-valued functions on γ. Under the suitable conditions, we establish analogues of Sokhotski–Plemelj formulas, obtain explicit solutions of RBVPs (jump, homogeneous, and non-homogeneous), and show that solvability and the number of linearly independent solutions are determined by the index $\varkappa$ of G.

This work was supported by a grant from the Simons Foundation (SFI-PD-Ukraine-00014586, S.A.P., R.P.P.) The second author was also supported by the budget program "Support for the development of priority areas of research" (KPKVK 6541230).

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On Carathéodory boundary extension of mappings with moduli inequality

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All definitions and notions used below may be found in [1]. Let $A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$, let $B(x_0, r_0) = \{x \in \mathbb{R}^n : |x - x_0| < r_0\}$, let $M_p(\cdot)$ be the modulus of family of paths of the order $p \geqslant 1$, and let $dm(y)$ be an element of the Lebesgue measure in $\mathbb{R}^n$, $n \geqslant 2$. Given sets E, $F \subset \overline{\mathbb{R}^n}$ and a domain $D \subset \mathbb{R}^n$ we denote by $\Gamma(E, F, D)$ a family of all paths $\gamma : [a, b] \to \overline{\mathbb{R}^n}$ such that $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for $t \in (a, b)$. If $f : D \to \mathbb{R}^n$, $y_0 \in f(D)$ and $0 < r_1 < r_2 < d_0 = \sup\limits_{y \in f(D)} |y - y_0|$, then by $\Gamma_f(y_0, r_1, r_2)$ we denote the family of all paths γ in D such that $f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$. Let $Q : \mathbb{R}^n \to [0, \infty]$ be a Lebesgue measurable function. We say that f *satisfies Poletsky inverse inequality* at the point $y_0 \in \overline{f(D)}$, if the relation

$$M(\Gamma_f(y_0, r_1, r_2)) \leqslant \int\limits_{A(y_0, r_1, r_2) \cap f(D)} Q(y) \cdot \eta^n(|y - y_0|)\, dm(y) \tag{1}$$

holds for any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that $\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1$.

Theorem 1. *Let D and D' be domains in $\mathbb{R}^n$, $n \geqslant 2$, and let D be a domain with a weakly flat boundary. Suppose that f is open discrete mapping of D onto D' satisfying the relation (1) at each point $y_0 \in \overline{D'}$. In addition, assume that the following conditions are fulfilled:*

1) for each point $y_0 \in \partial D' \setminus \{\infty\}$ there is $0 < r_0 \leqslant \sup\limits_{y \in D'} |y - y_0|$ such that, for any $0 < r_1 < r_2 < r_0$ there exists a set $E_1 \subset [r_1, r_2]$ of positive linear Lebesgue measure such that Q is integrable on $S(y_0, r)$ for $r \in E_1$;

2) $C(f, \partial D) \subset E$ for some set $E \subset \overline{D'}$ which is closed in $\overline{\mathbb{R}^n}$, while D' is locally finitely connected with respect to E, in other words, for each point $z_0 \in E$ and for any neighborhood U of this point there exists a neighborhood $V \subset U$ of z_0 such that the set $V \cap (D' \setminus E)$ consists of a finite number of components;

3) the set $f^{-1}(E \cap D')$ is nowhere dense in D.

Then the mapping f has a continuous extension $\overline{f} : \overline{D} \to \overline{D'}$, moreover, $\overline{f}(\overline{D}) = \overline{D'}$.

This result is published in [2].

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Acknowledgements. This talk is supported by the National Research Foundation of Ukraine (Project "Analogues of Carathéodory and Koebe-Bloch theorems for Orlycz-Sobolev classes", Project number 2025.02/0010).

Estimates of the modulus of continuity of the logarithmic double layer potential in the closure of domain

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For the real part of the Cauchy-type integral, which is known to be the logarithmic double layer potential, we obtain estimates of the modulus of continuity in the closure of domain bounded by an Ahlfors-regular integration curve. These estimates are more exact than the well-known Zygmund estimate for the modulus of continuity of the Cauchy-type integral. The accuracy of estimates for the logarithmic double layer potential is proved by constructing an example of a curve and an integral density for which the specified estimates are exact with respect to the order of smallness.

The talk is dedicated to the memory of Professor Oleg Gerus from Zhytomyr, Ukraine.

The work is supported by the National Research Foundation of Ukraine, Project number 2025.07/0014, Project name "Modern problems of Mathematical Analysis and Geometric Function Theory".

Berezin and Odzijewicz-type quantizations of arbitrary smooth manifolds

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We exhibit Berezin and Odzijewicz-type quantizations on arbitrary smooth manifolds by embedding them in $\mathbb{C}P^n$ or $\mathbb{C}^n$ and inducing the Poisson structure and the quantizations from the ambient manifold. If time permits we talk about some applications.

On equicontinuity of mappings with branching defined in family of domains

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All definitions and notions used below may be found in [1]. Let $S_i = S(x_0, r_i) = \{x \in \mathbb{R}^n : |x - x_0| = r_i\}$, $i = 1, 2$, $n \geqslant 2$, and let $Q : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable function satisfying the condition $Q(x) \equiv 0$ for $x \in \mathbb{R}^n \setminus D$. In what follows, $M_p(\cdot)$ denotes the modulus of family of paths of the order $p \geqslant 1$, and $dm(x)$ is an element of the Lebesgue measure in $\mathbb{R}^n$. Let $A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$. Given $p \geqslant 1$, a mapping $f : D \to \overline{\mathbb{R}^n}$ is called a *ring Q-mapping at a point $x_0 \in \mathbb{R}^n$ with respect to p-modulus*, if the condition

$$M_p(f(\Gamma(S_1, S_2, D))) \leqslant \int\limits_{A \cap D} Q(x) \cdot \eta^p(|x - x_0|)\, dm(x) \tag{1}$$

holds for all $0 < r_1 < r_2 < \infty$ and any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1\,. \tag{2}$$

Below $h(x, y)$ denotes the chordal (spherical metric) between points $x, y \in \overline{\mathbb{R}^n}$. Let $h(E)$ be achordal diameter of the set E in $\overline{\mathbb{R}^n}$. For $x_0 \in \mathbb{R}^n$ and $r_0 > 0$, as usual, $B(x_0, r_0) = \{x \in \mathbb{R}^n : |x - x_0| < r_0\}$. Given $p \geqslant 1$, a domain $D_0 \subset \mathbb{R}^n$, a set $E \subset \overline{\mathbb{R}^n}$ and a number $\delta > 0$, a fixed domain $D_0 \subset \mathbb{R}^n$, $n \geqslant 2$, a sequence of domains $\mathfrak{D} = \{D_m\}_{m=1}^{\infty}$ the kernel of which is D_0 and a Lebesgue measurable function $Q : \mathbb{R}^n \to [0, \infty]$ we denote by $\mathfrak{R}_{Q,\delta,p,E}(D_0, \mathfrak{D})$ the family of all open discrete closed mappings $f : D_m \to \overline{\mathbb{R}^n} \setminus E$ satisfying (1)–(2) for any $0 < r_1 < r_2 < \infty$ at any point $x_0 \in \overline{D_0}$ such that the following condition holds: for any domain $D_m^{\prime} := f_m(D_m)$ there exists a continuum $K_m \subset D_m^{\prime}$ such that $h(K_m) \geqslant \delta$ and $h(f_m^{-1}(K_m), \partial D_m) \geqslant \delta > 0$. The following statement holds.

Theorem 1. *Let $p \in (n-1, n]$ and let $f_m \in \mathfrak{R}_{Q,\delta,p,E}(D_0, \mathfrak{D})$, $m = 1, 2, \ldots,$ be a sequence such that:*

1) the sequence of domains D_m is regular with respect to D_0;

2) for every $m \in \mathbb{N}$, a domain D_m is locally connected on its boundary;

3) the family $f_m(D_m)$ is equi-uniform with respect to p-modulus over all $m \in \mathbb{N}$;

4) at least one of two following conditions hold: Q has a finite mean oscillation in $\overline{D_0}$, or

$$\int\limits_{0}^{\beta(x_0)} \frac{dt}{t^{\frac{n-1}{p-1}} {q_{x_0}^{\prime}}^{\frac{1}{p-1}}(t)} = \infty \tag{3}$$

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for every $x_0 \in \overline{D_0}$ *and some* $\beta(x_0) > 0$, *where* $q'_{x_0}(t)$ *denotes the integral average of the function* Q' *under the sphere* $S(x_0, t)$, *while* $Q'(x) = \begin{cases} Q(x), & Q(x) \geqslant 1\,, \\ 1, & Q(x) < 1\,. \end{cases}$ *Let* E *be a set of a positive capacity for* $p = n$, *and* E *is arbitrary closed set for* $n - 1 < p < n$.

I. *Then the family* f_m, $m = 1, 2, \ldots$, *is uniformly equicontinuous in* $\mathfrak{D}$, *i.e., for any* $\varepsilon > 0$ *there is* $\delta = \delta(\varepsilon) > 0$ *such that* $h(f_m(x), f_m(y)) < \varepsilon$ *whenever* $x, y \in D_m$, $|x - y| < \delta$ *and* $m \in \mathbb{N}$. *Moreover, there is a subsequence* f_{m_k}, $k = 1, 2, \ldots$, *which converges to* f *locally uniformly in* D_0. *In this case,* f *has a continuous boundary extension* $f : \overline{D_0} \to \overline{\mathbb{R}^n}$. *If* $x_m \in D_m$, $m = 1, 2, \ldots$, f_m *converges to* f *locally uniformly in* D_0 *and* $x_m \to x_0$ *as* $m \to \infty$, *then* $f_m(x_m) \to f(x_0)$.

II. *In addition, assume that*

5) $p = n$ *and there is* $r > 0$, *does not depending on* $m \in \mathbb{N}$, *such that* $h(E) \geqslant r$ *whenever* E *is any component of* $\partial f_m(D_m)$. *If* A_0 *is some compactum in* D_0, *then there is* $\delta_2 > 0$ *and* $M_0 \in \mathbb{N}$ *such that* $A_0 \subset D_m$ *and* $h(f_m(A_0), \partial f_m(D_m)) \geqslant \delta_2$ *for every* $m \geqslant M_0$. *The mapping* f *is boundary preserving: if* $x_0 \in \partial D_0$, *then* $f(x_0) \in \partial f(D_0)$. *If* $\overline{B(x_0, \varepsilon_0)} \subset D_0 \cap \bigcap\limits_{m=1}^{\infty} D_m$, *then there is* $\varepsilon_1 > 0$ *does not depending on* $m \in \mathbb{N}$ *such that*

$$B_h(f_m(x_0), \varepsilon_1) \subset f_m(D_m)\,, \quad m = 1, 2, \ldots\,, \tag{4}$$

where $B_h(y_0, r_0) = \{y \in \overline{\mathbb{R}^n} : h(y, y_0) < r_0\}$.

The mentioned result is obtained in [2].

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Acknowledgements. This talk is supported by the National Research Foundation of Ukraine (Project number 2025.07/0014, Project name: "Modern problems of Mathematical Analysis and Geometric Function Theory".).

On Carathéodory theorem for Orlicz-Sobolev classes

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All definitions and notions used below may be found in [1]. The following notation is used: the set of prime ends corresponding to the domain D is denoted by E_D, and the completion of the domain D by its prime ends is denoted by $\overline{D}_P$. Below $C(f, \partial D)$ denotes the cluster set of f on ∂D. The following result holds.

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Theorem 1. *Let $n-1<\alpha\leqslant n$, let D and D' be bounded domains in $\mathbb{R}^n$, $n\geqslant 3$, let $Q:D\to[0,\infty]$ be a Lebesgue measurable function and let $\varphi:[0,\infty)\to[0,\infty)$ be an increasing function. Let D be a regular domain and let $f:D\to D'$ be an open discrete mapping in $W^{1,\varphi}_{\rm loc}(D)$, $f(D)=D'$. In addition, assume that $C(f,\partial D)\subset E_*$ for some closed (in the topology of $\mathbb{R}^n$) set $E_*\subset\overline{D'}$ and $f^{-1}(E_*)=E$ for some closed (in the topology of D) subset $E\subset D$. In addition, assume that: 1) the set E is nowhere dense in D and D is finitely connected on $E\cup\partial D$, i.e., for any $z_0\in E\cup\partial D$ and any neighborhood $\widetilde{U}$ of z_0 there is a neighborhood $\widetilde{V}\subset\widetilde{U}$ of z_0 such that $(D\cap\widetilde{V})\setminus E$ consists of finite number of components, 2) for any $P\in E_D:=\overline{D}_P\setminus D$ and for every neighborhood U of P in $\overline{D}_P$ there is a neighborhood $V\subset U$ in $\overline{D}_P$ of P such that $V\cap D$ is connected and $(V\cap D)\setminus E$ consists at most of m components, $1\leqslant m<\infty$, 3) all components of the set $D'\setminus E_*$ have a strongly accessible boundary with respect to α-modulus, 4) the function φ satisfies the following Calderon condition*

$$\int\limits_1^{\infty}\left(\frac{t}{\varphi(t)}\right)^{\frac{1}{n-2}}\,dt<\infty\,. \tag{1}$$

5) Assume that $K_{I,\alpha}(x,f)\leqslant Q(x)$ a.e. whenever $K_{I,\alpha}(x,f)$ is the inner dilatation of f of the order α and, in addition, the condition $\int\limits_0^{\delta(b)}\frac{dt}{t^{\frac{n-1}{\alpha-1}}{q_b^{\prime}}^{\frac{1}{\alpha-1}}(t)}=\infty$ holds for every point $b\in\partial D$ and some $\delta(b)>0$, where $q_b^{\prime}(t)$ denotes the integral average of the function Q' under the sphere $S(b,t)$, while $Q'(x)=\begin{cases}Q(x), & Q(x)\geqslant 1\,,\\ 1, & Q(x)<1\,.\end{cases}$ Then f has a continuous extension $\overline{f}:\overline{D}_P\to\overline{D'}$, moreover, $\overline{f}(\overline{D}_P)=\overline{D'}$.

This result is published in [2].

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Acknowledgements. This talk is supported by the National Research Foundation of Ukraine (Project "Analogues of Carathéodory and Koebe-Bloch theorems for Orlycz-Sobolev classes", Project number 2025.02/0010).

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On Koebe-Bloch theorem under some integral constraints

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All necessary definitions may be found in [1]. Let $A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$, let $B(x_0, r_0) = \{x \in \mathbb{R}^n : |x - x_0| < r_0\}$, let $M_p(\cdot)$ be the modulus of family of paths of the order $p \geqslant 1$, and let $dm(y)$ be an element of the Lebesgue measure in $\mathbb{R}^n$, $n \geqslant 2$. We say that f *satisfies the inverse Poletsky inequality at a point* $y_0 \in \overline{f(D)} \setminus \{\infty\}$ *with respect to p-modulus,* if the relation

$$M_p(\Gamma_f(y_0, r_1, r_2)) \leqslant \int\limits_{A(y_0,r_1,r_2)\cap f(D)} Q(y) \cdot \eta^p(|y - y_0|)\, dm(y) \tag{1}$$

holds for any $0 < r_1 < r_2 < r_0 < \infty$ and any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that $\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1$. Given $p \geqslant 1$, a non-decreasing function $\Phi\colon \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$, $a, b \in D$, $a \neq b$, $\delta > 0$ we denote by $\mathfrak{F}^{\Phi,p}_{a,b,\delta}(D)$ the family of all open discrete mappings $f : D \to \mathbb{R}^n$, $n \geqslant 2$, for which there exists a Lebesgue measurable function $Q = Q_f : \mathbb{R}^n \to [0, \infty]$ satisfying relations (1) at any point $y_0 \in \overline{\mathbb{R}^n}$ with $\int\limits_{\mathbb{R}^n} \Phi(Q_f(y)) \cdot \frac{dm(y)}{(1+|y|^2)^n} < \infty$ such that $h(f(a), f(b)) \geqslant \delta$, where h is a chordal metric in $\overline{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty\}$. The following statement holds.

Theorem 1. *Let $n \geqslant 2$, $p \in (n-1, n]$ and let D be a domain in $\mathbb{R}^n$. Let also $\Phi : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ be an increasing convex function that satisfies the condition $\int\limits_{\delta}^{\infty} \frac{d\tau}{\tau(\Phi^{-1}(\tau))^{\frac{1}{p-1}}} = \infty$ for some $\delta > \Phi(0)$. Assume that, the family $\mathfrak{F}^{Q,p}_{a,b,\delta}(D)$ is equicontinuous at a and b. Then for every compactum K in D and for every $0 < \varepsilon < \operatorname{dist}(K, \partial D)$ there exists $r_0 = r_0(\varepsilon, K) > 0$ which does not depend on f, such that $f(B(x_0, \varepsilon)) \supset B_h(f(x_0), r_0)$ for all $f \in \mathfrak{F}^{\Phi,p}_{a,b,\delta}(D)$ and all $x_0 \in K$, where $B_h(f(x_0), r_0) = \{w \in \overline{\mathbb{R}^n} : h(w, f(x_0)) < r_0\}$.*

The mentioned result is established in [2].

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Acknowledgements. This talk is supported by the National Research Foundation of Ukraine (Project "Analogues of Carathéodory and Koebe-Bloch theorems for Orlycz-Sobolev classes", Project number 2025.02/0010).

The explicit form of the approximate solution to the Boltzmann equation for the hard-sphere model

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The Boltzmann kinetic equation plays an important role in the kinetic theory of gases. It is describes the evolution of rarefied gases. For the hard sphere model, the equation has the form [1]

$$D(f) = Q(f, f); \tag{1}$$

$$D(f) \equiv \frac{\partial f}{\partial t} + \left(v, \frac{\partial f}{\partial x}\right), \tag{2}$$

$$Q(f, f) \equiv \frac{d^2}{2} \int_{R^3} dv_1 \int_{\Sigma} d\alpha |(v - v_1, \alpha)| \Big[f(t, x, v_1') f(t, x, v') - f(t, x, v) f(t, x, v_1) \Big]. \tag{3}$$

The only exact solution to equation (1), which is known explicitly up to now, is the Maxwell distribution M or simply Maxwellian (after J. C. Maxwell, Scottish physicist). It makes both parts of the Boltzmann equation equal to zero, namely

$$D(M) = 0, \quad Q(M, M) = 0. \tag{4}$$

The solution to this equation (1)-(3) will be look for in the next form[2]

$$f(t, x, v) = \sum_{i=1}^{\infty} \varphi_i(t, x) M_i(t, x, v), \tag{5}$$

where coefficient functions $\varphi_i(t, x)$ are nonnegative smooth functions on $\mathbb{R}^4$. M_i are Maxwellians (4), which describe the "acceleration-compression" type of motion of the gas.

As a measure of the deviation between the parts of equation (1) we will consider a uniform-integral error of the form:

$$\Delta = \sup_{(t,x) \in \mathbb{R}^4} \int_{\mathbb{R}^3} \big| D(f) - Q(f, f) \big| dv. \tag{6}$$

In the paper [2], we were obtained sufficient conditions for the coefficient functions and hydrodynamic parameters appearing in the distribution, which enable one to make the analyzed error (6) as small as desired.

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Structure of gradient flows with a degenerate node on the sphere with holes

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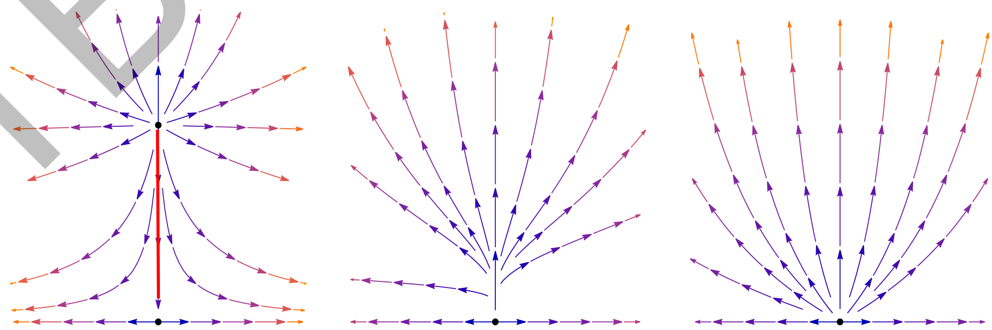
There are two types of degenerated nodes on the boundary that arise at $a = 0$ for such families of vector fields:

1) $V_1(x, y, a) = \{x, y^2 + ay\}$, $y \geq 0$, if the node is a source (HN_+ bifurcation, see Fig. 0.1);
2) $V_2(x, y, a) = \{-x, -y^2 - ay\}$, $y \geq 0$, if the node is a sink (HN_- bifurcation).

Here x, y are local coordinates, a is a real parameter.

In Fig. 0.1 a) the vector field before the bifurcation is shown ($a = -1$), in Fig. 0.1 b) – the vector field at the moment of bifurcation ($a = 0$), and in Fig. 0.1 c) – the vector field after the bifurcation ($a = 1$).



(a) (b) (c)

FIGURE 0.1. HN_+ bifurcation

Similar bifurcation can happen only on the boundary of a compact surface. In particular, both before and after the bifurcation the vector fields are Morse vector fields. A flow generated by the vector field in the moment of bifurcation is called an *HN-flow*.

We research construction of complete topological invariants for flows generated by vector fields before the moment of bifurcation — Morse flows, as a matter of fact. It is a *distinguishing graph* which is a generalization of a *chord diagram* represented in our previous work [1].

The algorithm of construction is made as follows. We take a Morse flow on a sphere with a finite number of holes, highlight all the stable manifolds of saddle points of the flow and mark all the boundary trajectories as well. Then, we cut out small enough neighbourhoods of all the sources in this flow. We obtain a specific discrete topological graph in which we have just to colour singular points (vertices of the graph) and trajectories (edges of the graph):

1) its edges representing boundaries of neighbourhoods of sources have a second colour (they are dashed), its edges representing stable manifolds of saddles in the interior of the sphere have a first (thin black) colour, the other edges have a third (thick black) colour;
2) the sinks of the flow lying on the boundary of the sphere have a white colour and the other vertices have a black colour.

Theorem 1. *HN-flows are topologically equivalent if and only if their distinguishing graphs are isomorphic; moreover, that isomorphism preserves orientations of the trajectories and colours of all the vertices and the edges.* [2]

Also the code for such flows was constructed. It is a specific finite sequence of symbols which corresponds to a certain distinguishing graph and thus represents a certain HN-flow with regard to a homeomorphism.

Finally, all the HN-flows on the sphere with one, two and three holes were counted provided that the number of all singular points of the flow is at most 6. So, for instance, there are 6 HN_+-flows with 5 singular points and 24 HN_+-flows with 6 singular points on D^2.

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The equations of Gauss, Codazzi and Ricci of surfaces in 4-dimensional space forms

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Let N be a Riemannian, neutral or Lorentzian 4-dimensional space form with constant sectional curvature L_0. We can understand the equations of Gauss, Codazzi and Ricci of a space-like or time-like surface in N, in terms of the induced connection (of the complexification) of the two-fold exterior power of the pull-back bundle on the surface.

Suppose that N is Riemannian. Then the two-fold exterior power of the pull-back bundle on a surface in N is decomposed into two orientable subbundles of rank 3. Computing the curvature tensors of these subbundles, we obtain the expressions of the equations of Gauss, Codazzi and Ricci of the surface given in [8] ([7]). By the equations of Gauss and Ricci, the degeneracy of the twistor lifts of a surface just means that the surface satisfies $K \equiv L_0$ for the intrinsic curvature K and $R^\perp \equiv 0$ for the curvature tensor $R^\perp$ of the normal connection ([7]). We obtain a characterization of a surface such that the covariant derivative of the complex lift by $\partial/\partial w$ or $\partial/\partial \overline{w}$ for every local complex coordinate w of the surface vanishes ([7]). The twistor lifts of such a surface are degenerate.

Suppose that N is neutral. Then analogous discussions and results are valid for space-like or time-like surfaces in N. We obtain a characterization of a time-like surface in N with $L_0 \neq 0$ which has local sections of the time-like twistor spaces with fully light-like covariant derivative on a neighborhood of each point of the surface ([7]). Refer to [2], [4], [5], [6] for time-like surfaces with zero mean curvature vector such that their time-like twistor lifts have zero or light-like covariant derivative. We obtain characterizations of a time-like surface in N such that the time-like twistor lifts are horizontal along integral curves of a light-like one-dimensional distribution ([7]). In particular, a time-like surface has this property if and only if the surface has zero mean curvature vector so that the shape operator of any normal vector field is zero or light-like, and therefore its time-like twistor lifts have zero or light-like covariant derivative ([4]). We obtain a characterization of a time-like surface in N such that the covariant derivative of the paracomplex lift by $\partial/\partial \check{w}$ or

$\partial/\partial\overline{\check{w}}$ for every local paracomplex coordinate $\check{w}$ of the surface vanishes ([7]). If the mean curvature vector of such a surface vanishes, then a light-like normal vector field is parallel ([7]), and therefore its time-like twistor lifts have zero or light-like covariant derivative ([4]). The time-like twistor lifts of the above surfaces are degenerate.

Suppose that N is Lorentzian. Then analogous discussions and results are also valid for space-like or time-like surfaces in N. In this case, we need the decomposition of the complexification of the two-fold exterior power of the pull-back bundle on a space-like or time-like surface, and the Levi-Civita connection of N gives connections of the two subbundles ([7]). The complex twistor spaces associated with the pull-back bundle are fiber bundles over the surface in the two subbundles. The complex twistor lifts of the surface are sections of the complex twistor spaces. We obtain a characterization of a space-like surface in N such that the covariant derivative of a complex twistor lift by $\partial/\partial w$ or $\partial/\partial\overline{w}$ for every local complex coordinate w of the surface vanishes ([7]). For a space-like surface with zero mean curvature vector, this property just means that a light-like normal vector field is parallel (refer to [1], [3], [8] for characterizations of such a surface). Moreover, if such a surface has no totally geodesic points, then it is strictly isotropic in the sense of [3] ([7]). We obtain characterizations of a time-like surface in N such that a complex twistor lift is horizontal along integral curves of a light-like one-dimensional distribution, and in particular, such a surface has zero mean curvature vector so that the shape operator of any normal vector field is zero or light-like ([7]). Refer to [8] for characterizations of such a surface based on [3]. The complex twistor lifts of the above surfaces are degenerate.

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Quadric surfaces of revolution as cubic Weingarten surfaces

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While the geometry of quadric surfaces of revolution is well-established within Euclidean space, their behavior and classification within the three-dimensional sphere $\mathbb{S}^3$ have remained largely under-examined. This presentation addresses this theoretical oversight by introducing a systematic taxonomy of non-degenerate spherical quadrics—specifically ellipsoids, hyperboloids, and paraboloids. These are analyzed as surfaces generated by the revolution of spherical conics around geodesic axes that are either incident to their foci or oriented orthogonally to them.

By utilizing spherical angular momentum as a primary geometric invariant, this study identifies these quadrics as a distinguished category of Weingarten surfaces (cf. [4]). We demonstrate that these surfaces are uniquely determined by a specific cubic functional relation between their principal curvatures. Furthermore, these results reveal an unexpected structural consistency across different ambient spaces; the cubic Weingarten relations identified in $\mathbb{S}^3$ directly parallel those previously documented in both Euclidean and Lorentzian frameworks (see [1], [2] and [3]). This suggests a fundamental universality in the curvature dynamics of quadric surfaces regardless of the underlying ambient manifold's signature or curvature.

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A Class of Functionals on the Sequence Space s Satisfying the Palais-Smale Condition

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In this talk, we introduce a class of functionals on the space of rapidly decreasing sequences s, called $\mathcal{F}_s$-functionals, defined as decomposable sums of quadratic and convex terms with quadratic growth. We prove that such functionals satisfy the Palais–Smale condition and admit a unique global minimum. Furthermore, we show that the Palais–Smale condition is preserved under linear homeomorphisms. This allows us to construct corresponding functionals satisfying the Palais–Smale condition on Fréchet spaces isomorphic to the space s. We then show how this framework provides a tool for the formulation and proof of existence and uniqueness of solutions for specific nonlinear operator problems in function spaces.

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On some properties of affinor-semisymmetric spaces of class one

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Consider a pseudo-Riemannian space V_n of class one with a metric tensor $g_{ij}(x)$, in which an affinor $F_i^h(x)$ is defined. The tensor characteristics of class one spaces take the form

$$R_{hijk} = e(b_{hk}b_{ij} - b_{hj}b_{ik}), \quad b_{ij,k} - b_{ik,j} = 0, \tag{1}$$

where b_{ij} is the second fundamental form of the space V_n, and $e = \pm 1$. For spaces of class one, the following holds:

$$R_{ij} = e(bb_{ij} - b_{i\alpha}b_j^{\alpha}), \quad b \stackrel{def}{=} b_{\alpha\beta}g^{\alpha\beta} \tag{2}$$

Suppose that the affinor structure defined in this space is antisymmetric, semisymmetric, and almost complex, i.e., such that it satisfies

$$F_{ij,[kl]} = 0, \quad F^h_{\cdot\alpha}F^{\alpha}_{\cdot i} = -\delta_i^h, \quad F_{(ij)} = 0. \tag{3}$$

Spaces for which the first condition holds will be called affinor-semisymmetric. In spaces with an affinor structure, the operation of conjugation is defined. For such spaces, from the integrability conditions of the affinor, taking into account the Ricci identity, it follows that

$$R_{ij\overline{kl}} = R_{ijkl}. \tag{4}$$

Multiply the identity (1) by b_l^h, sum over the index h, perform the index substitution $l \to h$, and conjugate with respect to the indices j, k

$$b_{h\alpha}R^{\alpha}_{\cdot i\bar{j}\bar{k}} = bR_{hi\bar{j}\bar{k}} + R_{h\bar{j}}b_{i\bar{k}} - R_{h\bar{k}}b_{i\bar{j}}, \tag{5}$$

or, taking into account (4)

$$R_{hj}b_{ik} - R_{hk}b_{ij} - R_{h\bar{j}}b_{i\bar{k}} + R_{h\bar{k}}b_{i\bar{j}} = 0. \tag{6}$$

Contracting this expression over the indices i, k, we obtain

$$R_{hj} = \frac{R}{2b}(b_{hj} + b_{\bar{h}\bar{j}}). \tag{7}$$

The integrability conditions of an curvature tensor, taking into account the Ricci identity, take the form

$$R_{hijk,[\bar{l}\bar{p}]} = R_{hijk,[lp]}. \tag{8}$$

Therefore, substituting (1) into this expression and contracting first with respect to h, l, and then with respect to p, k, we obtain $RR_{ij} = 0$. From this, we obtain that either the Ricci tensor $R_{ij} = 0$ or the scalar curvature $R = 0$, but zero scalar curvature is a consequence of the Ricci tensor being zero. Moreover, the converse also holds. Indeed, if $R = 0$, substituting this into expression (7) yields $R_{ij} = 0$. Thus, we have proved:

Theorem 1. *In affinor-semisymmetric spaces of class one, the Ricci tensor is equal to zero.*

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Disjoint F-semi-transitivity in Banach algebras

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Motivated by the concept of disjoint topological transitivity for supercyclicity, in this talk, we will consider the concept of disjoint Furstenberg-semi-transitivity for operators that are a composition of an isometric isomorphism and a left multiplier on a normed algebra. Thus, we will characterize disjoint $\mathcal{F}$-semi-transitive and disjoint supercyclic such operators on a large class of non-unital normed algebras. It turns out that generalized weighted bilateral shifts on the standard Hilbert C*-module are just a special case of our theory. Generalized weighted composition operators on the normed algebra of operator-valued continuous functions vanishing at infinity on a locally compact, non-compact Hausdorff space are another special case of our theory. Next, we will characterize disjoint $\mathcal{F}$-semi-transitive and disjoint supercyclic weighted composition operators on a large class of weighted solid Banach function spaces and we apply our results to the case of translations on weighted Morrey spaces. Finally, we will illustrate all the results with concrete examples. The results are generalizations and extensions of the results from [1, 2, 3].

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Lie-like structures over a product of vector spaces, with application to the Sheffer and Riordan groups

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The Riordan group was introduced in paper [6], and its study therein was significantly based upon the previous results from the umbral calculus, e.g. [5]. While the original motivation for defining the Riordan group came from enumerative combinatorics, this group also found applications in matrix theory, group theory, number theory, mathematical physics, computer science and other research areas in mathematical sciences. We refer to the monograph [7], solely devoted to the Riordan group, its extensions, deformations and applications.

The Riordan group consists of infinite lower triangular matrices $P=[p_{ni}]_{n,i=0,1,2,\dots}$ whose matrix elements p_{ni} satisfy the condition $\sum_{n=0}^{\infty} p_n(x)\xi^n = A(\xi)\big(1-xB(\xi)\big)^{-1}$, where $p_n(x)=\sum_{i=0}^{n} p_{ni}\,x^i$, and $A(\xi)$ and $B(\xi)$ are formal power series, which depend on the matrix P. This definition provides alternative realisations of the Riordan group as one consisting of polynomial sequences $(p_n(x))_{n=0}^{\infty}$, or one consisting of pairs $(A(\xi), B(\xi))$ of formal power series. The product in the Riordan group is the usual product of infinite lower diagonal matrices. This product can be explicitly described as a certain group product of pairs of formal power series $(A(\xi), B(\xi))$. In particular, the Riordan group contains the subgroup of pairs $(1, B(\xi))$, in which the group product is the composition of formal power series. It should be noted that such a group of formal power series was introduced in the 1950s [3], and has been actively studied since then as an algebraic, geometric and topological object.

A close relative of the Riordan group is the Sheffer group (also called the exponential Riordan group). This group consists of infinite lower triangular matrices P for which the corresponding polynomial sequence $(p_n(x))_{n=0}^{\infty}$ (called a Sheffer sequence) has the exponential generating function $\sum_{n=0}^{\infty} \frac{1}{n!} p_n(x)\xi^n = A(\xi) \exp\big(xB(\xi)\big)$. The Sheffer polynomial sequences play an important role in probability, in particular, in the theory of Lévy processes, with applications found in mathematical finance.

The seminal paper [4] initiated rigorous studies of infinite-dimensional Lie groups and their Lie algebras. Such Lie groups and Lie algebras play an important role in mathematical physics. For example, the Lie group of local diffeomorphisms (with composition of diffeomorphisms as group operation) and its Lie algebra of vector fields are important for quantum physics. The study of infinite-dimensional Lie groups and Lie algebras remains an active research area.

In papers [1, 2], both the Riordan group and the Sheffer group have been studied as infinite-dimensional Lie groups and their Lie algebras have been identified, including the explicit form of the Lie bracket. In particular, it was shown in [2] that the Lie algebra of the Sheffer group consists of the following linear operators acting in polynomials: $\sum_{k=1}^{\infty} a_k D^k + X \sum_{k=2}^{\infty} b_k D^k$, where D is the operator of differentiation, X is the operator of multiplication by the variable of the polynomials, and a_k, b_k are constants. Since the operators X and D satisfy the commutation relation $DX - XD = 1$, they generate a Weyl algebra, which is fundamental for quantum field theory. In particular, the explicit form of the Lie bracket in the Lie algebra of the Sheffer group can be seen as a consequence of the commutation relation in the Weyl algebra [2].

The concepts of the Riordan group and the Sheffer group and the study of their Lie structures were extended to the multivariate case, and even to the case of infinitely many variables, cf. [7, Chapter 7] and [2]. Sheffer polynomials on an infinite-dimensional space play an important role in infinite-dimensional analysis, quantum probability and mathematical physics.

To apply Milnor's theory [4] to study Lie structures of the Sheffer and Riordan groups over an infinite-dimensional space in [2], some crucial assumptions were made about the topological structure of the underlying space.

The aim of this talk is to show that it is possible to develop a Lie-like theory of the Sheffer and Riordan groups over an arbitrary vector space, without any use of topology. This allows us not only to significantly simplify the arguments used in [2] but also extend them to a much wider class of underlying vector spaces, which are important for applications.

This is a joint work with Nouf Alghamdi (Swansea University).

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Exponential Tail Estimates for Lacunary Trigonometric Series

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We establish explicit non-asymptotic upper bounds for the tail probabilities of normalized lacunary sums of the form

$$\nu_N(x) = \frac{f_N(x)}{\sigma(N)}$$

where

$$f_N(x) := \sum_{k=1}^{N} c_k \sqrt{2}\cos(2\pi n_k x), \quad x \in [0,1],$$

$\{n_k\}$ is a strictly increasing sequence of integers satisfying the Hadamard gap condition:

$$n_{k+1} \ge q\, n_k, \quad q > 1, \quad \forall\, k \ge 1,$$

$\{c_k\}$ is a sequence of real numbers not square-summable, i.e.,

$$\sum_{k=1}^{\infty} c_k^2 = \infty,$$

$\sigma^2(N)$ is the variance of f_N defined by

$$\sigma^2(N) := \mathrm{Var}(f_N) = \mathbb{E}[f_N^2] - (\mathbb{E}[f_N])^2,$$

$\mathbb{E}[f_N]$ is the expectation given by

$$\mathbb{E}[f_N] = \int_0^1 f_N(x)\, dx.$$

The variance satisfies

$$\sigma^2(N) = \sum_{k=1}^{N} c_k^2.$$

Since the variance diverges, the normalized sums exhibit non-trivial behavior.

In particular, we have

$$\mathbb{E}[\nu_N] = 0, \qquad \mathrm{Var}(\nu_N) = 1.$$

Classical asymptotic results describe the behavior of ν_N as $N \to \infty$, while we provide explicit bounds for fixed N. More precisely, for fixed N, explicit exponential estimates are obtained for the tail probabilities

$$T[\nu_N](t) = \mathbb{P}\{x \in [0,1] : \nu_N(x) \ge t\}, \qquad t > 0,$$

in terms of the lacunarity parameter q and the normalizing factor $\sigma(N)$.

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Furthermore, some examples are provided to illustrate the obtained bounds for different choices of coefficients c_k, highlighting the transition from subgaussian to stretched-exponential tail behavior.

This study is motivated by several applications, including Fourier series, signal processing, and the analysis of stochastic processes, where explicit non-asymptotic bounds for the tail probabilities of partial sums play an important role.

The results presented here are contained in [1].

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Stratified Hamiltonian Flows on Immersions of the Projective Plane

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Let M be the image of an immersion $g\colon \mathbb{RP}^2 \to \mathbb{R}^3$, considered as a stratified space whose strata are determined by the singular set of the immersion.

A *simple stratified Hamiltonian flow (SSH-flow)* on M is an low satisfying the following conditions:

- the flow has no singular points on the one-dimensional strata;
- for each two-dimensional stratum, flow is the image of the Hamiltonian flow on a region of S^2 and the corresponding Hamiltonian function is a simple Morse function;
- in a neighborhood of every zero-dimensional stratum, the restrictions of the vector field to the adjacent one-dimensional strata are locally generated by Morse functions.

There exist two non-homeomorphic surfaces that are immersions of the projective plane with one triple point and a set of double points with three components – Boy's surface and Girl's surface [1]. The topological structure of flows with gradient dynamics on these surfaces was described in [2, 3]. The study of the structure of simple SH-flows is similar to the study of pro-Hamiltonian flows on non-orientable surfaces [4, 5].

Theorem 1. *On Boy's and Girl's surfaces, there exist 5 distinct structures of simple stratified Hamiltonian flows without internal saddle points.*

The qualitative structure of such flows can be described combinatorially by means of oriented Reeb graphs enriched with additional data associated with the singular strata.

The *distinguishing graph* of a simple SH-flow is defined as its oriented Reeb graph together with the cyclic ordering of the edges adjacent to the vertices corresponding to singular strata of the immersion.

Two distinguishing graphs are said to be equivalent if there exists a graph isomorphism preserving:

- the orientation of all edges;
- the types of vertices;

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- the cyclic order of the edges incident to the distinguished vertices.

Theorem 2 (Criterion of Topological Equivalence)**.** *Simple stratified Hamiltonian flows are trajectory topologically equivalent if and only if their distinguishing graphs are equivalent.*

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On solutions of the matrix equation $AX + YB = C$ over an elementary divisor domain

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Let R be an elementary divisor domain with identity $e \neq 0$. We introduce the following notations: $\mathrm{R}_{m,n}$ is the set of $m \times n$ matrices over R, I_n is the identity $n \times n$-matrix, $0_{m,n}$ is the zero $m \times n$-matrix and D^* is the classical adjoint matrix for a nonsingular matrix $D \in \mathrm{R}_{n,n}$, i.e., $D^*D = DD^* = (\det D)I_n$.

In an elementary divisor domain R all finitely generated ideals are principal. So, for two given elements $a, b \in \mathrm{R}$ there exists a greatest common divisor $d = (a, b)$ and elements $x, y \in \mathrm{R}$ satisfying the Bezout identity $ax + by = d$ (see [4]). Thus, every matrix A over an elementary divisor domain R admits a diagonal reduction, that is, if $A \in \mathrm{R}_{m,n}$ and $\mathrm{rank}A = r$, then there exist invertible matrices $P \in GL(m, \mathrm{R})$ and $Q \in GL(n, \mathrm{R})$ such that $U_A A V_A = S_A = \mathrm{diag}(a_1, a_2, \dots, a_r, 0, \dots, 0)$ is a diagonal matrix, where $a_1, a_2, \dots, a_r$ are all nonzero and $a_i | a_{i+1}$ (divides) for all $i = 1, 2, \dots, r-1$. The matrix S_A is called the Smith normal form of the matrix A.

Consider the matrix equation

$$AX + YB = C, \tag{1}$$

where $A \in \mathrm{R}_{m,m}$, $B \in \mathrm{R}_{n,n}$ and $C \in \mathrm{R}_{m,m}$ be given matrices and let $X, Y \in \mathrm{R}_{m,n}$ be unknown matrices. Solving matrix equation (1) over fields and commutative rings is one of the most important research problems, which has applications in many areas of mathematics and its applications.

Roth [10] proved that equation (1) is solvable over a field $\mathrm{R} = \mathbb{F}$ if and only if matrices $M_C = \begin{bmatrix} A & C \\ 0_{n,m} & B \end{bmatrix}$ and $M_0 = \begin{bmatrix} A & 0_{m,n} \\ 0_{n,m} & B \end{bmatrix}$ are equivalent. It is easy to verify that equation (1) is compatible over a field if and only if $\mathrm{rank}M_0 = \mathrm{rank}A + \mathrm{rank}B$.

Jones [6] proved that Roth's compatibility condition for equation (1) is true for the ring of analytic functions over the field of complex numbers $\mathbb{C}$. Gustafson in [5] proved that equation (1) over a commutative ring K with unity is compatible if and only if matrices M_C and M_0 are equivalent over K. If $\mathrm{K} = \mathbb{F}[\lambda]$ is the ring of polynomials over a field $\mathbb{F}$, then in works [1], [2], [3], [7], [8], [9], and [11] conditions were investigated under the matrix equation $A(\lambda)X(\lambda) - Y(\lambda)B(\lambda) =$

$C(\lambda)$ over $\mathbb{F}[\lambda]$ admits "minimal" solutions over $\mathbb{F}[\lambda]$, i.e. solutions that $\deg X(\lambda) < \deg B(\lambda)$ (or $\deg Y(\lambda) < \deg A(\lambda)$).

We note that the review of the literature and comments on current research is given in the extended paper [12]. In this report we investigate the solvability of the Sylvester-type matrix equation $AX + YB = C$ over an elementary divisor domain.

Theorem 1. *Let $A \in \mathrm{R}_{m,m}$ and $B \in \mathrm{R}_{n,n}$ be nonsingular matrices and $(\det A, \det B) = d$. Then for arbitrary matrix $C \in \mathrm{R}_{m,n}$ the matrix equation*

$$AX + YB = dC \tag{2}$$

is solvable over R. *Further, let $u, v \in \mathrm{R}$ such that $u(\det A) + v(\det B) = d$. Then for arbitrary $t \in \mathrm{R}$ and for arbitrary matrix $P \in \mathrm{R}_{m,n}$ the pair of matrices*

$$X = A^*C(u + t(\det B)) + A^*P(\det B) \quad \text{and} \quad Y = (v - t(\det A))CB^* - (\det A)PB^*$$

is the general solution of equation (2).

Corollary 2. *Let $A \in \mathrm{R}_{m,m}$ and $B \in \mathrm{R}_{n,n}$ be nonsingular matrices and $C \in \mathrm{R}_{m,n}$. Further let $(\det A, \det B) = d$. If $C = dC_1$ then the matrix equation $AX + YB = C$ is solvable over* R.

Further, let $u, v \in \mathrm{R}$ such that $u(\det A) + v(\det B) = d$. Then for arbitrary $t \in \mathrm{R}$ and for arbitrary matrix $P \in \mathrm{R}_{m,n}$ the pair of matrices

$$X = A^*C_1(u + t(\det B)) + A^*P(\det B) \quad \text{and} \quad Y = (v - t(\det A))C_1B^* - (\det A)PB^*$$

is the general solution of equation $AX + YB = C$.

Corollary 3. *Let $A \in \mathrm{R}_{m,m}$ and $B \in \mathrm{R}_{n,n}$ be nonsingular matrices and $(\det A, \det B) = e$. Then for arbitrary matrix $C \in \mathrm{R}_{m,n}$ the matrix equation $AX + YB = C$ is solvable over* R.

Further, let $u, v \in \mathrm{R}$ such that $u(\det A) + v(\det B) = e$. Then for arbitrary $t \in \mathrm{R}$ and for arbitrary matrix $P \in \mathrm{R}_{m,n}$ the pair of matrices

$$X = A^*C(u + t(\det B)) + A^*P(\det B) \quad \text{and} \quad Y = (v - t(\det A))CB^* - (\det A)PB^*$$

is the general solution of equation $AX + YB = C$.

Proposition 4. *Let $A \in \mathrm{R}_{m,m}$ and $B \in \mathrm{R}_{n,n}$ be nonsingular matrices and $C \in \mathrm{R}_{m,n}$. Further, let $(\det A, \det B) = d$. If the matrix equation $AX + YB = C$ is solvable over* R *then* $A^*CB^* = 0(\mod d)$.

Theorem 5. *Let $A \in \mathrm{R}_{m,m}$ and $B \in \mathrm{R}_{n,n}$ be nonsingular matrices and $C \in \mathrm{R}_{m,n}$. Further, let $S_A = \mathrm{diag}(e, \ldots, e, a)$ and $S_B = \mathrm{diag}(e, \ldots, e, b)$ be the Smith normal forms of matrices A and B respectively and $(\det A, \det B) = d$. If $A^*CB^* = 0(\mod d)$ then matrix equation $AX + YB = C$ is solvable over* R.

Problem 6. When are conditions of Proposition 4 sufficient for the existence of solutions of the equation $AX + YB = C$ over an elementary divisor domain R ?

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On F-planar mappings of quasi-Kähler spaces

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A Riemannian space (V_n, g_{ij}, F_i^h) with metric tensor g_{ij} and affine structure F_i^h is called *quasi-Kähler*([3]) if the conditions

$$F_\alpha^h F_i^\alpha = -\delta_i^h,$$

$$g_{i\alpha} F_j^\alpha = -g_{j\alpha} F_i^\alpha,$$

$$F_{i,j}^h = -F_{\alpha,\beta}^h F_i^\alpha F_j^\beta,$$

where the comma «,» is the sign of the covariant derivative with respect to the connection of V_n. Quasi-Kähler spaces include well-known classes of almost complex manifolds, such as Kähler, K-, H-spaces.

It is easy to prove that a quasi-Kähler structure is integrable if and only if it is Kähler.

Consider the quasi-Kähler spaces (V_n, g_{ij}, F_i^h) and $(\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h)$. In the general coordinate system (x^i) the F-planar mapping ([2])

$$(V_n, g_{ij}, F_i^h) \longrightarrow (\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h),$$

is characterized by the following basic equations:

$$\overline{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x) + \psi_{(i}\delta_{j)}^h + \phi_{(i}F_{j)}^h,$$

$$F_i^h(x) = \overline{F}_i^h(x),$$

$$F_\alpha^h F_i^\alpha = -\delta_i^h,$$

$$F_{ij} = -F_{ji}, \qquad \overline{F}_{ij} = -\overline{F}_{ji},$$

$$F_{ij} = g_{i\alpha} F_j^\alpha, \qquad \overline{F}_{ij} = \overline{g}_{i\alpha} F_j^\alpha,$$

$$F_{i,j}^h = -F_{\alpha,\beta}^h F_i^\alpha F_j^\beta, \qquad F_{i|j}^h = -F_{\alpha|\beta}^h F_i^\alpha F_j^\beta,$$

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where $\Gamma_{ij}^h, \overline{\Gamma}_{ij}^h$ are the Christoffel symbols $V_n, \overline{V}_n$, respectively; $\psi_i(x)$, $\phi_i(x)$ are some covectors; brackets (i, j) denote the symmetrization operation; comma «,» and vertical bar «|» are the signs of the covariant derivative with respect to the connection of V_n and $\overline{V}_n$,, respectively.

An F-planar mapping is considered trivial if $\psi_i = \phi_i = 0$.

We have proved that for a nontrivial F-planar mapping, the vectors ψ_i and ϕ_i are related:

$$\psi_{\bar{i}} = -\phi_i, \qquad \phi_{\bar{i}} = \psi_i.$$

We agree to denote the affinor contraction operation as follows:

$$B^{\dots}_{\bar{i}\dots} = B^{\dots}_{\alpha\dots} F^{\alpha}_{i}, \qquad B^{\bar{i}\dots}_{\dots} = B^{\alpha\dots}_{\dots} F^{i}_{\alpha}$$

A number of geometric objects (both inhomogeneous and tensor-like) invariant under F-planar mappings are constructed.

An F-planar mapping of a quasi-Kähler space onto a flat Riemannian space is considered.

Proved

Theorem 1. *In order for a quasi-Kähler space* (V_n, g_{ij}, F_i^h) *to admit an* F*-planar mapping onto a flat space, it is necessary that the Riemann tensor in it satisfies the conditions*

$$R^h_{ijk} - R^{\bar{h}}_{\bar{i}jk} + R^h_{i\bar{j}}\bar{k} - R^{\bar{h}}_{\bar{i}\bar{j}\bar{k}} = \frac{4R}{n(n+2)}\left[\delta^h_k g_{ij} - \delta^h_j g_{ik} - F^h_j F_{ik} + F^h_k F_{ij} + 2F^h_i F_{kj}\right].$$

We will call quasi-Kähler spaces whose Riemann tensor satisfies the specified conditions *generalized* F*-flat.*

Theorem 2. *The class of generalized* F*-flat quasi-Kähler spaces* (V_n, g_{ij}, F_i^h) *is closed under* F*-planar mappings.*

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Controllability problems for the parabolic equation in a half-plane controlled by the Dirichlet boundary condition with a bounded control

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Let $T > 0$, $\Omega = \big((0, +\infty) \times \mathbb{R}\big)$, $\Delta = (\partial/\partial x_1)^2 + (\partial/\partial x_2)^2$. Consider the following control system in a half-plane

$$w_t = \Delta w, \qquad (x_1, x_2) \in \Omega,\ t \in (0, T), \tag{1}$$

$$w\big(0, (\cdot)_{[2]}, t\big) = u((\cdot)_{[2]}, t), \qquad x_2 \in \mathbb{R},\ t \in (0, T), \tag{2}$$

$$w\big((\cdot)_{[1]}, (\cdot)_{[2]}, 0\big) = w^0, \qquad (x_1, x_2) \in \Omega, \tag{3}$$

where $u \in U[0, T]$ is a control,

$$U[0,T] = \left\{ \varphi \in L^\infty(\mathbb{R} \times (0,T)) \;\middle|\; \sup_{t \in [0,T]} |\varphi(\cdot, t)| \in L^2(\mathbb{R}) \right\}$$

is the set of admissible controls. The subscripts [1] and [2] associate with the variable numbers, e.g. $(\cdot)_{[1]}$ and $(\cdot)_{[2]}$ correspond to x_1 and x_2, respectively, if we consider $f(x)$, $x \in \mathbb{R}^2$. We investigate control system (1)–(3) in subspaces of Sobolev spaces $H^{-2s}(\mathbb{R}^2)$ with $s = 0, 1$. In particular, we consider $w(\cdot, t) \in L^2(\Omega)$, $t \in [0, T]$ and $w^0 \in L^2(\Omega)$. We study the approximate controllability problem for system (1)–(3).

Definition 1. A state $w^0 \in L^2(\Omega)$ is said to be *approximately controllable* to a target state $w^T \in L^2(\Omega)$ in a given time $T > 0$ if for each $\varepsilon > 0$, there exists $u_\varepsilon \in U[0, T]$ such that there exists a unique solution w_ε to system (1)–(3) with $u = u_\varepsilon$ and $\left\| w_\varepsilon\big((\cdot)_{[1]}, (\cdot)_{[2]}, T\big) - w^T \right\|_{L^2} < \varepsilon$.

Controllability problems for parabolic equations were studied both in bounded and unbounded domains. However, most of the papers studying these problems deal with domains bounded with respect to the spatial variables.

The case of a bounded domain is fully studied. It is well-known that each initial state of control system (1)–(3) in a bounded domain $D \subset \mathbb{R}^n$ with the boundary ∂D of class C^2 can be driven to zero for a given time $T > 0$. This result was obtained by using Carleman inequalities (see, e.g. [1]).

The case of unbounded domain is essentially different. An initial state cannot be driven to an arbitrary target state, but only to a specific one that satisfies a certain relation with the initial state. For example, it has been proved in [2] that there is no initial data in any Sobolev space of negative order that may be driven to zero in finite time.

In the most of papers, controls belonging to L^2 were considered. But in our problem, we show that such controls are not appropriate to consider approximate controllability property for $w^0 \in L^2(\Omega)$ and $w(\cdot, t) \in L^2(\Omega)$, $t \in [0, T]$. We consider the initial state $w^0 = 0 \in L^2(\Omega)$ and construct the control $u \in L^2(\mathbb{R} \times (0,T))$ with compact support by means of which $w^0 = 0$ is driven to the end state $w(\cdot, T)$ that does not belong to $L^2(\Omega)$. That is why we consider the narrower set of controls $U[0,T]$, i.e., we consider a specific subset of bounded controls in $L^2(\mathbb{R} \times (0,T))$. This condition guarantees that $w(\cdot, t) \in L^2(\Omega)$, $t \in [0,T]$, for any $w^0 \in L^2(\Omega)$.

We prove the following theorem.

Theorem 2. *Each state $w^0 \in L^2(\Omega)$ is approximately controllable to any target state $w^T \in L^2(\Omega)$ in a given time $T > 0$.*

In other words, each initial state $w^0 \in L^2(\Omega)$ of system (1)–(3) can be driven to an arbitrary neighbourhood of any target state $w^T \in L^2(\Omega)$ by choosing an appropriate control $u \in U[0,T]$.

The method of proving this theorem is constructive and it provides us a numerical algorithm of constructing of controls solving the approximate controllability problem for system (1)–(3) and of approximate end states which are reached by means of these controls. We consider the odd extension of w and w^0 with respect to x_1 and obtain a new control system. Then we reduce the controllability problem for the 2-d parabolic equation controlled by the Dirichlet boundary condition to a finite family of controllability problems for the 1-d ones controlled by the same boundary conditions. For this we develop the state and the control in a new control system in the Fourier series with respect to a basis generated by Hermite functions. For solving the approximate controllability problem for the 1-d parabolic equation, we apply the method introduced in [3]. As a result we construct controls solving our problem and approximate end states which are reached by means of constructed controls. Finally, we analyse the solution to control problem by applying the Fourier transform and its inverse.

The theorem and the numerical algorithm of solving the approximate controllability problem for system (1)–(3) are illustrated by an example.

All obtained results are published in [4].

Acknowledgements. The authors are partially supported by the Akhiezer Foundation and "Pauli Ukraine Project", funded in the WPI Thematic Program "Mathematics–Magnetism–Materials" (2024/2025).

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AGMA in the method of proving Minkowski's conjecture on Diophantine approximations, its applications, and generalizations

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We demonstrate methods from algebra, geometry, and mathematical analysis (AGMA) applied to prove results on Diophantine approximations, packings, and coverings related to Minkowski's conjecture on critical determinant.

We investigate Minkowski balls, Minkowski spheres and Minkowski domains on the real plane. Minkowski spheres are boundaries of Minkowski balls. We consider balls of the form

$$D_p: \ |x|^p + |y|^p \leq 1, \ p \geq 1,$$

and call such balls *Minkowski balls*. Continuing this, we consider the following classes of Minkowski balls and circles (one-dimensional spheres).

- *Watson balls*: $|x|^p + |y|^p \leq 1$ for $p_0 > p \geq 2$;
- *Davis balls*: $|x|^p + |y|^p \leq 1$ for $p_0 > p \geq 2$;
- *Mordell-Chebyshev balls*: $|x|^p + |y|^p \leq 1$ for $p \geq p_0$;

Remark 1. This classification corrects the classification given in [3] and more accurately reflects the historical contributions of researchers.

Remark 2. Minkowski domains $2^m D_p$ are 2^m doubled Minkowski balls.

The algebraic part of the study concerns free modules, which are algebraic images of geometric lattices, and their invariants. This part (for brevity, we use lattice language), like geometry and analysis, is presented in theorems.

Theorem 3. *Let $m \in \mathbb{N}$ (see [5] for the case $m = 1$). The critical determinants of 2^m dubling balls D_p have a representation of the form*

$$\Delta_p^{(0)}(2^m D_p) = \Delta(p, \sigma_p)_{2^m D_p} = 2^{2m-1} \cdot \sigma_p, \ \sigma_p = (2^p - 1)^{1/p}, \tag{1}$$

$$\Delta_p^{(1)}(2^m D_p) = \Delta(p, 1)_{2^m D_p} = 4^{m - \frac{1}{p}} \frac{1 + \tau_p}{1 - \tau_p}, \ 2(1 - \tau_p)^p = 1 + \tau_p^p, \ 0 \leq \tau_p < 1. \tag{2}$$

And these are the determinants of the sublattices of index 2^m of the critical lattices of the corresponding balls D_p.

The extremal functions for packing of Minkowski balls and domains have the forms:

$$\Delta(2^m D_p) = \begin{cases} \Delta_p^{(1)}(2^m D_p), \ 1 < p \leq 2, \ p \geq p_0, \\ \Delta_p^{(0)}(2^m D_p), \ 2 \leq p \leq p_0; \end{cases} \tag{3}$$

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Quantum models of the Riemann zeta function, algebraic entanglement models, and (adelic) Galois representations

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Hilbert-Polya conjecture connects the Riemann Hypothesis on the Riemann zeta function with the eigenvalues of quantum Hamiltonians. This also applies to Selberg's trace formula and Selberg classes. In this presentation, we plan to continue and expand on the discussions of the preprint [4]. Selected current problems in quantum mathematics related to these problems will be briefly reviewed, based on spectral methods and the adelic approach. Algebraic entanglement models and (adelic) Galois representations will also be presented.

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Subwreath product as structure of normal subgroups of symmetric group wreath product

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In this research, we continue our previous investigation of wreath product normal structure [1, 2] Normal subgroups and their structures for finite and infinite iterated wreath products $S_{n_1} \wr \ldots \wr S_{n_m}$, $n, m \in \mathbb{N}$ and $A_n \wr S_n$ are founded.

Let $k(\pi)$ be the number of cycles in the decomposition of a permutation π of degree n.

The number $n - k(\pi)$ is denoted by $dec(\pi)$, and is called a decrement [6] of permutation π. As well known, [6] the minimal number of transpositions in factorization of a permutation π on transpositions happens to be equal to $dec(\pi)$. We set $dec(e) = 0$. If $\pi_1, \pi_2 \in S_n$, then the following formula holds:

$$dec(\pi_1 \cdot \pi_2) = dec(\pi_1) + dec(\pi_2) - 2m, m \in \mathbf{N}, \quad (1)$$

Definition 1. The permutational *subwreath product* $G \wr\wr H$ is the semi-direct product $G \ltimes \tilde{H}^X$, where G acts on the subdirect product [4] $\tilde{H}^X$ by the respective permutations of the subdirect factors. Provided the specification of $\tilde{H}^X$ is established separately.

Definition 2. The set of elements from $S_n \wr S_n, n \geqslant 3$ which is presented by the Kaloujnine tableaux [3] of the form: $[e]_0,\ [a_1, a_2, \ldots, a_n]_1$, satisfying the following condition

$$\sum_{i=1}^{n} dec([a_i]_1) = 2k, k \in \mathbb{N}, \tag{2}$$

be denoted by $\widetilde{A}_n^{(1)}$. Note that condition (2) uniquely identifies subdirect product.

We spread the definition of $\widetilde{A}_n^{(1)}$ on the 3-multiple wreath product by a recursive way.

Also we prove that $\widetilde{A}_n^{(1)}$ is the **normal subgroup** of $S_n \wr S_n$ so it will be called ***alternating level subgroup*** $\widetilde{A}_n^{(1)}$ with structure $E \wr\wr \widetilde{A}_n$, where $\widetilde{A}_n$ is the subdirect product of n copies of S_n uniquely identified by (2).

Theorem 3. *The subgroup $\widetilde{A}_n^{(1)}$ has **normal rank** $n-1$ in $S_n \wr S_n$, $n \geqslant 3$, $n \cong 1(mod2)$ and **normal rank** n iff $n \geqslant 3$, $n \cong 0(mod2)$.*

Definition 4. *The subgroup $E \wr \widetilde{A}_n^{(1)}$ be denoted by $\widetilde{A}_n^{(2)}$.*

Furthermore, we prove that $E \wr \widetilde{A}_n^{(2)} \lhd S_n \wr S_n \wr S_n$. The order of $E \wr \widetilde{A}_n^{(2)}$ is $(n!)^{3n} : 2^3$. The subgroup $\widetilde{A}_n^{(1)}$ has **normal rank** 2 [7] in $S_n \wr S_n$.

Definition 5. The set of elements from $S_n \wr S_n \wr S_n, n \geqslant 3$ presented by the tables [3] form: $[e]_0,\ [e, e, \ldots, e]_1, [a_1, a_2, \ldots, a_n]_2$, satisfying the following condition

$$\sum_{i=1}^{n} dec([a_i]_2) = 2k, k \in \mathbb{N}, \tag{3}$$

be denoted by $\widetilde{A}_{n^2}^{(2)}$. Note that condition (3) uniquely identifies subdirect product in $\prod_{i=1}^{n^2} S_n$ as base of subwreath product, the similar subdirect product describing commutator of wreath product was investigated by us in [8] in research of pronormality it appears in [9].

Proposition 6. *The subgroup $\widetilde{A}_n^{(1)} \lhd S_n \wr S_n$ as well as $\widetilde{A}_n^{(2)} \lhd S_n \wr S_n \wr S_n$. Furthermore $\widetilde{A}_n^{(2)} \lhd \widetilde{A}_{n^2}^{(2)}$.*

Definition 7. *A subgroup in $S_n \wr S_n$ is called $\widetilde{T_n}$ if it consists of:*

(1) elements of $E \wr A_n$,
(2) elements with the tableau [3] presentation $[e]_1,\ [\pi_1, \ldots, \pi_n]_2$, that $\pi_i \in S_n \setminus A_n$.

One easy can validates a correctness of this definition, i.e. that the set of such elements form a subgroup and its normality. This subgroup has structure

$$\tilde{T}_n \simeq (\underbrace{A_n \times A_n \times \cdots \times A_n}_{n}) \rtimes C_2 \simeq \underbrace{S_n \boxplus S_n \ldots \boxplus S_n}_{n},$$

where the operation $\boxplus$ of a subdirect product is subject of item 1) and 2)

Remark 8. The order of $\widetilde{T_n}$ is $\frac{(n!)^n}{2^{n-1}}$.

Definition 9. The unique minimal normal subgroup is called the monolith.

Theorem 10. *The monolith of $S_n \wr S_m$ is $e \wr A_m$.*

Theorem 11. *Proper normal subgroups in $S_n \wr S_m$, where $n, m \geq 3$ with $n, m \neq 4$ are of the following types:*

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(1) subgroups that act only on the second level are

$$E \wr\wr \widetilde{A}_m,\ \widetilde{T}_m,\ E \wr S_m, E \wr A_m,$$

(2) subgroups that act on both levels are $A_n \wr\wr \widetilde{A}_m,\ S_n \wr\wr \widetilde{A}_m,\ A_n \wr S_m$, *wherein the subgroup* $S_n \wr\wr \widetilde{A}_m \simeq S_n \leftthreetimes (\underbrace{S_m \boxtimes S_m \boxtimes S_m \ldots \boxtimes S_m}_{n})$ *endowed with the subdirect product satisfying to condition* (2).

Theorem 12. *The full list of normal subgroups of* $W = S_n \wr S_n \wr S_n$ *consists of 50 normal subgroups. These subgroups are the following:*

1. ***Type*** T_{023} *contains:* $E \wr\wr \tilde{A}_n \wr\wr H$, $\widetilde{T_n} \wr H$, *where* $H \in \{\tilde{A}_n, \tilde{A}_{n^2}, S_n\}$. *There are 6 subgroups.*
2. ***The second type of subgroups is subclass in*** T_{023} *with a new base of wreath product subgroup* $\tilde{A}_{n^2}$*:* $E \wr S_n \wr\wr \tilde{A}_{n^2}$, $E \wr N_i(S_n \wr S_n)$. *Therefore, this class has 12 new subgroups. Thus, the total number of normal subgroups in **Type*** T_{023} *is 18.*
3. ***Type*** T_{003}*:* $A^{(3)}_{00(n^2)}$, $\widetilde{T}_{n^2}$, $\widetilde{T_n}^{(3)}$.
4. ***Type*** T_{123}*:* $N_i(S_n \wr S_n) \wr S_n$, $N_i(S_n \wr S_n) \wr \tilde{A}_n$ *and* $N_i(S_n \wr S_n) \wr\wr \tilde{A}_{n^2}$. *Thus, there are 29 new normal subgroups in* T_{123}, *taking into account a repetition.*

Remark 13. Note that $E \wr \widetilde{A}^{(1)}_n \simeq E \wr (E \wr\wr \widetilde{A}_n)$ contains in the family $E \wr N_i(S_n \wr S_n)$.

The condition of invariance of subgroups of H with depth l [2] of a permutation wreath product W is presented in the following theorem.

Definition 14. The set of elements from $\mathop{\wr}\limits_{i=0}^{k} S_{n_i}, n_i \geqslant 3$ with depth k satisfying the following condition

$$\sum_{i=(s-1)n^m+1}^{sn^m} dec([a_i]_k) = 2t, t \in \mathbb{N},\ 1 \le s \le n^{k-m},\ \ [a_i]_j = e,\ for\, j = \overline{0, k-1} \qquad (4)$$

be called $\widetilde{A}^{(k)}_{n^m}$, where $1 \le m < k$.

Theorem 15. *If* $H \lhd W$, *where* $d(H) = l$ *and* $l:\ k - l = m$, *then* $A^{(k)}_{n^m} \lhd H$.

We denote by $Aut_f X^*$ the group of all finite automorphism of spherically homogeneous rooted tree.

Theorem 16. *Let* $H \lhd Aut_f X^*$ *having depth* k, *then* H *contains* k*-th level subgroup* P *having all even vertex permutations* $p_{ki} \in A_n$ *on* X^k *and trivial permutations in vertices of rest of levels.*

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Weakly and strongly connected Markov graphs for maps on trees

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Let X be a finite combinatorial tree, and $\sigma\colon V(X) \to V(X)$ be a map from the vertex set of X to itself. The Markov graph $\Gamma(X, \sigma)$ is a directed graph having edges of X as its vertices with the existence of an arc $\{u, v\} \to \{x, y\}$ indicating the fact that $\{u, v\}$ "covers" $\{x, y\}$ under σ (meaning that x, y lie on a unique shortest path between $\sigma(u)$ and $\sigma(v)$ in X).

Recall that a digraph is called weakly connected provided the corresponding undirected graph is connected. The next results describe the structure of self-maps on trees with weakly connected Markov graphs (see [1]). We start with maps that produce weakly connected Markov graphs for some trees.

Proposition 1. *Let $n = |V| \geq 3$. Then for a map $\sigma\colon V \to V$, there exists a tree X on V such that $\Gamma(X, \sigma)$ is weakly connected if and only if σ is neither a constant map nor the identity map on V.*

A map $\sigma\colon V \to V$ is called nilpotent if for some $k \in \mathbb{N}$, the map σ^k is constant. A nilpotent map σ is path-nilpotent provided $|\sigma^{-1}(x) \setminus \{x\}| = 1$ for all $x \in \sigma(V)$. A permutation σ of V is called circular of length m if σ has exactly $|V| - m$ fixed points and it cyclically permutes its m non-fixed points. The next result characterizes maps that produce weakly connected Markov graphs for all trees.

Theorem 2. *Let $n = |V| \geq 2$. For a map $\sigma\colon V \to V$, the Markov graph $\Gamma(X, \sigma)$ is weakly connected for all trees X on V if and only if σ satisfies one of the following conditions:*

(1) σ is a path-nilpotent map;
(2) σ is a circular permutation of length $n - 1$;
(3) σ is a permutation, and the periods of σ-periodic points with disjoint orbits are pairwise incomparable under division.

Recall that a digraph is called strongly connected provided it has a spanning closed walk. From Theorem 2 it follows that every cyclic permutation on a tree produces a weakly connected Markov graph. The next theorem describes trees which admit cyclic permutations with non-strongly connected Markov graphs (see [2]). A forest is called balanced if each of its trees has the same number of vertices.

Theorem 3. *For a tree X, there exists a cyclic permutation σ of $V(X)$ such that its Markov graph $\Gamma(X, \sigma)$ is not strongly connected if and only if there exists a proper spanning set of edges $E' \subset E(X)$ such that $X[E']$ is a balanced forest.*

Proposition 4. *Let $n = |V| \geq 3$. For a map $\sigma\colon V \to V$, the Markov graph $\Gamma(X, \sigma)$ is strongly connected for all trees X on V if and only if n is a prime number and σ is a cyclic permutation.*

Finally, we describe the structure of permutations that produce strongly connected Markov graphs for some trees.

Theorem 5. *Let $n = |V| \geq 5$. For a permutation $\sigma\colon V \to V$, there exists a tree X on V such that $\Gamma(X, \sigma)$ is strongly connected if and only if σ has at most $\frac{n-1}{2}$ fixed points.*

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Conformal mappings and invariance of the Einstein equations

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Let ψ be a spinor field in an n-dimensional space-time $(V^{1,n-1}, g)$. The Belinfante–Rosenfeld stress-energy tensor of the spinor field calculated by the formula [3]

$$T_{jk} = \frac{i}{2}\big(\overline{\psi}\gamma_{(j}\nabla_{k)}\psi - (\nabla_{(j}\overline{\psi})\gamma_{k)}\psi\big). \tag{1}$$

It should satisfy the Einstein Equations [3]

$$R_{ij} - \frac{1}{2}Rg_{ij} = -8\pi G T_{ij}, \tag{2}$$

where g_{ij}, $R_{ij} = R^{\alpha}_{ij\alpha}$ and $R = R_{ij}g^{ij}$ are the metric, the Ricci tensor and the scalar curvature respectively of a space-time $(V^{1,n-1}, g)$. Obviously, R^{h}_{ijk} is the curvature tensor of $(V^{1,n-1}, g)$.

Let us consider conformal mapping $f : (V^{1,n-1}, g) \to (\tilde{V}^{1,n-1}, \tilde{g})$ i. e. diffeomorphism between the manifolds whose metrics are related as

$$\tilde{g}_{ij}(x) = e^{2\varphi(x)} g_{ij}(x),$$

where $\varphi(x)$ is a function of the x's. The following relations hold [6], [5]

$$\begin{aligned}
\tilde{R}^h_{ijk} &= R^h_{ijk} + \delta^h_k \varphi_{ij} - \delta^h_j \varphi_{ik} + g^{hl}(\varphi_{lk} g_{ij} - \varphi_{lj} g_{ik}) + (\delta^h_k g_{ij} - \delta^h_j g_{ik})\Delta_1 \varphi, \\
\tilde{R}_{ij} &= R_{ij} + (n-2)\varphi_{ij} + (\Delta_2 \varphi + (n-2)\Delta_1 \varphi) g_{ij}, \\
\tilde{R} &= e^{-2\varphi}(R + 2(n-1)\Delta_2 \varphi + (n-1)(n-2)\Delta_1 \varphi).
\end{aligned} \tag{3}$$

Here $\varphi_i = \partial_i \varphi$, $\Delta_1 \varphi = \varphi_i \varphi_j g^{ij}$, $\varphi_{ij} = \nabla_j \varphi_i - \varphi_i \varphi_j$, $\Delta_2 \varphi = \nabla_j \varphi_i g^{ij}$. We know that invariance of the Dirac equations requires that a spinor field should be transformed according to the formula [3]

$$\psi' = e^{s\varphi(x)} \psi.$$

The number $s = \frac{1-n}{2}$ is referred to as the conformal weight of a spinor field ψ.

Then the Belinfante–Rosenfeld stress-energy tensor (1) of a spinor field transformed within a conformal mapping according to the formula

$$\tilde{T}_{jk} = e^{(2s+1)\varphi} T_{jk}. \tag{4}$$

Note that for a massless spinor field [1], [2]

$$\tilde{T} = \tilde{T}_{jk} \tilde{g}^{jk} = T = T_{jk} g^{jk} = 0.$$

It follows from (2), (3) that the function φ must satisfy the system (massless case)

$$\nabla_j \varphi_i = \varphi_i \varphi_j - \frac{1}{2} g_{ij} \Delta_1 \varphi - \frac{8\pi G}{n-2} \left(\tilde{T}_{ij} - T_{ij} \right),$$

or taking into account (4)

$$\nabla_j \varphi_i = \varphi_i \varphi_j - \frac{1}{2} g_{ij} \Delta_1 \varphi - \frac{8\pi G T_{ij}}{n-2} \left(e^{(2s+1)\varphi} - 1 \right). \tag{5}$$

Conditions of integrability of (5) are [4]

$$\varphi_\alpha R^\alpha_{ijk} + \frac{8\pi G}{n-2} \Big(\big(\nabla_k T_{ij} - \nabla_j T_{ik} \big) \left(e^{(2s+1)\varphi} - 1 \right) + \big(\varphi_k T_{ij} - \varphi_j T_{ik} \big)(2s+1) e^{(2s+1)\varphi} \Big).$$

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On p-adic Differential Operator

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Abstract. Through a differential operator L, we study a “mod p differential system” $A(y) = 0$ on a disk D that is over a p-adic disk D^*. In this talk, we present the evaluation of the index of such a differential system. Also, we will measure the irregularity of such p-adic differential systems in a disk. This document proposes a method to connect local and global indices using the Mittag-Leffler theorem. The proposed method aims to enhance our understanding of the interplay between local solutions and global behavior in p-adic differential equations. These equations behave very differently from classical ones. This is mainly due to the non-archimedean nature of the field, which affects convergence and the behavior of solutions. As a result, special methods are required, especially when studying singular points and computing the index of operators. This study highlights the importance of understanding the unique properties of p-adic differential equations, particularly in relation to their singular points and irregularities in solutions.

Objectives:

- Explain how to calculate the index of a p-adic differential operator using complex analysis methods.
- Examine how regular and irregular singularities affect the indices.
- Analyze existing approaches such as the Adolhpson method and introduce new analytical tools.

MOTS-CLÉS p-adic diffrential operators, index, singularity.

Introduction

The p-adic differential operator, an important concept in functional and numerical analysis, plays a prominent role in the study of dynamical systems and differential equations in the non Archimedean sense. Calculating the index of a differential operator shows the number of independent solutions of a differential equation. This helps to better understand the operator, particularly if there is a singularity present. This study deals with the measurement of irregularity in a disk. The effect of regular and irregular singularities on the index of the operator is analyzed. The calculating the index, including Adolphson’s method, are considered and new, more precise and adaptable analytical techniques are proposed. In summary, our work aims to bridge local and global perspectives in p-adic differential equations, providing a framework for future research in this area.The interplay between local and global indices is crucial for understanding the structure of solutions in p-adic differential equations.

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On the Grothendieck embedding problem of linear closed subspaces into Banach spaces revisiting

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1. Introduction

The problems, concerned with estimation of the dimension of linear closed functional subspaces in $L_p\ (M, d\mu)$, $p > 1(\neq 2)$, (in part, in $L_p(0, 1; d\lambda)$, are of long time interest in analysis, being related with their many applications in operator and approximation theories, in dynamical systems theory and other applied fields. As an example one can recall here a central problem in Banach space theory to classify the complemented subspaces of $L_p(M, d\mu)$ up to isomorphism; the finite dimensional analogue is to find for any given $S_p \subset L_p(M, d\mu)$ a description of the finite dimensional spaces which are S_p-isomorphic to S_p-complemented subspaces of $L_p(M, d\mu)$. These problems were thoroughly studied before, in particular finite dimensional versions of this complemented subspaces of the $L_p(M, d\mu)$ problem, yet in neither case their classification is far to be closed.

It was also observed that sometimes happens that the finite dimensional version of an infinite dimensional problem leads to a theory [3, 4], giving rise to new interesting aspects of the infinite dimensional theory. Recall here the problem of describing the subspaces of $L_p(M, d\mu)$ which embed isomorphically into a "smaller" space, namely, the space l_p, for which there is found a fairly good answer. Assuming a space M is endowed with a probability measure μ, absolutely continuous with respect to a probabilistic measure ν, that is $d\mu = hd\nu$ regarding a strictly positive measurable function $h : M \to \mathbb{R}_+$ for which $\int_M hd\nu = 1$, we can induce for fixed $1 < p < \infty$ an isometry $J_h^{(p)}$ from $L_p(M, d\mu)$ onto $L_q(M, d\nu)$ for any $q > p > 1$. In particular, the next result, giving useful information about chosen *a priori* finite dimensional full support subspaces $S_p \subset L_p(M, d\mu)$.

Theorem 2. *Let μ be a probability measure on M and let S_p be a N-dimensional subspace of $L_p(M, d\mu)$, $1 < p < \infty$, with full support. Then there is a density $h > 0$ so that the image $J_h^{(p)} S_p$ has a basis $\{\varphi_1, \varphi_2, ..., \varphi_N\} \subset L_2(M, hd\mu)$, which is orthonormal in $L_2(M, hd\mu)$ and such that $\sum_{j=1}^N |\varphi_j|^2 = N$.*

Moreover, just recently there was announced an interesting result [2] concerning the Grothendieck type embedding theorem for Bergman spaces. Namely, let $A_p(\Omega) = Hol(\Omega) \cap L_p(\Omega)$ be a classical Bergman space of holomorphic functions on a bounded domain $\Omega \subset \mathbb{C}^n$, p - integrable on Ω with respect to the Lebesgue measure. Then the following theorem holds.

Theorem 3. *For any bounded domain $\Omega \subset \mathbb{C}^n$, if $1 \le p < q \le \infty$, E is a closed subspace of $A_p(\Omega)$ and $E \subset A_q(\Omega)$, then* $\dim E < \infty$.

Concerning a general problem of estimation the dimension of a linear closed topological subspace $S_p^{(q)} \subset L_p(M, d\mu) \hookrightarrow L_q(M, d\nu)$, identically embedded into the Banach space $L_q(M, d\nu), q > p + \alpha > 1(\neq 2)$, $\alpha \in (0, 1)$, we stated the following result.

Theorem 4. *Let a linear closed topological subspace $S_p^{(q)} \subset L_p(M, d\mu), p+\alpha > 1(\neq 2),$ be identically embedded into a Banach space $L_q(M, d\nu)$ for $q = 2 + (p + \alpha - 2)2^m > p + \alpha > 1(\neq 2), \alpha \in (0, 1),$ where $d\mu = hd\nu$ is a probability measure, absolutely continuous with respect to a probability measure $d\nu$ on M and satisfying the conditions $\int_M hd\nu = 1$ and $\int_M h^{(p+\alpha)/\alpha} d\nu < \infty$ for some $\alpha \in (0, 1)$. Then the dimension $\dim S_p^{(q)} = N \in \mathbb{N}$ of the closed subspace $S_p^{(q)} \subset L_p(M, d\mu)$ proves to satisfy the inequality $N \left(\frac{\Gamma(\frac{N}{2})\Gamma(\frac{q+1}{2})}{\sqrt{\pi}\Gamma(\frac{N+q}{2})} \right)^{2/q} \leq K^2_{p,q(\alpha,m)}$ for some fixed constant $K_{p,q(\alpha,m)} > 0$. If the degree $q \to \infty$, the dimension $\dim S_p^{(\infty)} \leq K^2_{p,q(\alpha,\infty)}$, being a priori finite.*

Remark 5. Taking into account the estimation of the dimension $\dim S_p^{(q)} = N \in \mathbb{N}$ of a linear closed subspace $S_p^{(q)} \subset L_p(M, d\mu) \hookrightarrow L_q(M, d\nu)$ for $q = 2+(p+\alpha-2)2^m > p+\alpha > 1(\neq 2), \alpha \in (0, 1),$ obtained in Theorem 4, it is interesting to analyse its interpretation and possible relationship to the known result in G. Pisier [3]:

Theorem 6. *In the space $L_2(0, 1)$ there exists a linear infinite-diemnsional closed subspace $G \subset L_2(0, 1)$, which is a closed subspace of every $L_p(0, 1), 1 \leq p < \infty$, and for which the corresponding norms are proportional, that is for arbitrary $1 \leq p < \infty$ there exist constants $\gamma_p > 0$, such that for any $g \in G$ the norms $||g||_p = \gamma_p ||g||_2$.*

In the case, when $\alpha \to 0$ and the probability measures $d\mu, d\nu$ on the set M are equal, the following corollary holds.

Let a linear closed topological subspace $S_p^{(q)} \subset L_p(M, d\mu), p > 1(\neq 2),$ be identically embedded into a Banach space $L_q(M, d\nu)$ for $q = 2 + (p - 2)2^m > p > 1(\neq 2)$, where $d\mu$ is a probability measure on M. Then the dimension $\dim S_p^{(q)} = N \in \mathbb{N}$ of the closed subspace $S_p^{(q)} \subset L_p(M, d\mu)$ proves to satisfy the inequality $N \left(\frac{\Gamma(\frac{N}{2})\Gamma(\frac{q+1}{2})}{\sqrt{\pi}\Gamma(\frac{N+q}{2})} \right)^{2/q} \leq K^2_{p,q(0,m)}$ for some fixed constant $K_{p,q(0,m)} > 0$. If the degree $q \to \infty$, the dimension $\dim S_p^{(\infty)} \leq K^2_{p,q(0,\infty)}$, being *a priori* finite.

As a consequence of the obtained results for a closed subspace in the Banach space $C([0, 1]; \mathbb{R})$ of continuous functions, closely imbedded into $L_q(0, 1; d\mu), q > 1$, we have stated the following Grothendieck type proposition.

Proposition 7. *Let $S_c^{(q)} \subset C([0, 1]; \mathbb{R})$ be a closed subspace of the Banach space $(C([0, 1]; \mathbb{R}), ||\cdot||_\infty)$ of continuous functions on the interval $[0, 1] \subset \mathbb{R}_+$, which allows the identical embedding into a Banach space $L_q(0, 1; d\nu), q > 1$, with respect to a probability measure $d\nu$ on $[0, 1]$. Then the subspace $S_c^{(q)} \subset C([0, 1]; \mathbb{R})$ is finite-dimensional.*

Acknowledgements: The authors are much indebted to A. M. Plichko, Ya. V. Mykytyuk and G. Pisier for both instructive discussions of diverse aspects of the work and useful remarks, comments and suggestions.

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On topologization of the extended bicyclic semigroup

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Every Hausdorff shift-continuous topology on the (extended) bicyclic semigroup is discrete [1, 3, 4]. When T_1-topology on the bicyclic monoid is discrete are given in [2].

Example 1. On the extended bicyclic semigroup $\mathcal{C}_{\mathbb{Z}}$ the following non-discrete T_1-topologies are constructed:

(i) a semigroup inverse topology;
(ii) a Baire left-continuous (right-continuous) topology;
(iii) a locally compact left-continuous (right-continuous) topology.

Proposition 2. *On the extended bicyclic semigroup $\mathcal{C}_{\mathbb{Z}}$ does not exist a countably compact left-continuous (right-continuous) T_1-topology.*

We give conditions under which a T_1-topology on the extended bicyclic semigroup $\mathcal{C}_{\mathbb{Z}}$ is discrete.

Theorem 3. *A left-continuous (right-continuous) T_1-topology τ on $\mathcal{C}_{\mathbb{Z}}$ is discrete if and only if there exists a sequence $\{(i_n, j_n)\}_{n\in\mathbb{N}}$ of isolated points in $(\mathcal{C}_{\mathbb{Z}}, \tau)$ such that the sequence $\{i_n\}_{n\in\mathbb{N}}$ $(\{j_n\}_{n\in\mathbb{N}})$ is strictly increasing.*

Theorem 4. *A shift-continuous T_1-topology τ on $\mathcal{C}_{\mathbb{Z}}$ is discrete if and only if the space $(\mathcal{C}_{\mathbb{Z}}, \tau)$ contains an isolated point.*

A topological space X is called *quasi-regular* at $x \in X$ if for any open neighbourhood $U(x)$ of x in X there exists an open nonempty subset V of X such that $\mathrm{cl}_X(V) \subseteq U(x)$.

Theorem 5. *If the T_1-space of a topological inverse semigroup $(\mathcal{C}_{\mathbb{Z}}, \tau)$ has a point x such that $(\mathcal{C}_{\mathbb{Z}}, \tau)$ is quasi-regular at x, then τ is discrete.*

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On topological properties of weakly $(n-1)$-convex sets in low-dimensional Euclidean spaces

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We consider topological and geometric properties of generalized convex sets in the real n-dimensional Euclidean space $\mathbb{R}^n$, where $n = 2, 3$, known as weakly $(n-1)$-convex sets. An open set in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) is called weakly 1-convex (resp. weakly 2-convex) if through every boundary point of the set there passes a straight line (resp. a plane) not intersecting the given set. A closed set in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) is called weakly 1-convex (resp. weakly 2-convex) if it is approximated from the outside by a family of open weakly 1-convex (resp. weakly 2-convex) sets. A point in the complement of a set in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) is a 1-nonconvexity (resp. 2-nonconvexity) point of the set if every straight line (resp. every plane) passing through the point intersects the set.

It is shown that each connected component of the 1-nonconvexity-point set corresponding to a planar weakly 1-convex set is the interior of a convex polygon or the interior of a convex generalized polygon [1]. It is proved that the 2-nonconvexity-point set corresponding to an open weakly 2-convex set in $\mathbb{R}^3$ is itself open and weakly 2-convex [1], that the non-empty interior of a solid closed weakly 2-convex set is weakly 2-convex [2]. A number of convexity properties of the 2-nonconvexity-point sets corresponding to weakly 2-convex sets in $\mathbb{R}^3$ are also investigated. In particular, It is shown that for any convex polyhedron there exists an open weakly 2-convex set such that its 2-nonconvexity-point set coincides with the polyhedron interior, however, there exists an open weakly 2-convex set such that its open bounded convex 2-nonconvexity-point set differs from the interior of a convex polyhedron [2]. Nevertheless, not every open convex set can occur as a connected component of the 2-nonconvexity-point set corresponding to some open weakly 2-convex set [1].

The author was supported by the National Research Foundation of Ukraine, Project number 2025.07/0014, Project name "Modern problems of Mathematical Analysis and Geometric Function Theory".

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On $R-$ Curvature Tensor of Locally Conformal $C_{12}-$ Manifolds

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Here are the names of (almost all) predefined theorem-like environments.

Definition 1. [3] An almost contact metric manifold $(M^{2n+1}, g, \xi, \eta, \Phi)$ is said to be a locally conformal $C_{12}-$manifold, if the following identity holds:

$$\begin{aligned}\nabla_X(\Phi)Y = &-g(X,\Phi Y)\alpha^{\#} + g(X,Y)\Phi(\alpha^{\#}) + \mathbf{d}\alpha(\Phi Y)X - \mathbf{d}\alpha(Y)\Phi X\\ &-\eta(X)\left\{\eta(Y)\left[\Phi(\nabla_\xi\xi) + \Phi(\alpha^{\#})\right] + g(\nabla_\xi\xi, \Phi Y)\xi + g(\alpha^{\#}, \Phi Y)\xi\right\},\end{aligned}$$

for all $X, Y \in X(M)$, where $\alpha^{\#} = grad\ \alpha$ is the vector field described by $g(\alpha^{\#}, X) = X(\alpha) = \boldsymbol{d}\alpha(X)$.

Theorem 2. [1] *Let $\{\omega^i\}$ be a local orthonormal coframe defined on the open set $U \subset M$; then on U there exist unique 1-forms $\{\theta^i_j\}$, $i, j = 0, 1, ..., 2n$, such that the second structure equation given by:*

$$\mathbf{d}\theta^i_j = -\theta^i_k \wedge \theta^k_j + \frac{1}{2}R^i_{jkl}\ \omega^k \wedge \omega^l.$$

Proposition 3. *If M^{2n+1} is a locally conformal $C_{12}-$manifold, then on M we have:*

1) $\theta^{\hat{a}}_b = (\alpha^b\delta^a_c - \alpha^a\delta^b_c)\ \omega^c$; 2) $\theta^b_{\hat{a}} = (\alpha_b\delta^c_a - \alpha_a\delta^c_b)\ \omega_c$;

3) $\theta^a_0 = C^a\ \omega + \alpha_0\ \omega^a$; 4) $\theta^{\hat{a}}_0 = C_a\ \omega + \alpha_0\ \omega_a$;

5) $\theta^0_a = -C_a\ \omega - \alpha_0\ \omega_a$; 6) $\theta^0_{\hat{a}} = -C^a\ \omega - \alpha_0\ \omega^a$,

where $a, b, c = 1, 2, ..., n$, and $\hat{a} = a + n$.

Theorem 4. *The Riemannian curvature tensor of locally conformal C_{12-}manifold M^{2n+1}, with $n > 1$ has the following components on G-structure adjoined space:*

(1) $R^a_{0\hat{h}0} = C^{ah} - C^aC^h$,
(2) $R^a_{0h0} = C^a{}_h - C^aC_h - \alpha_{00}\ \delta^a_h - (\alpha_0)^2\delta^a_h - 2\alpha^{[a}\delta^{d]}_h C_d$,
(3) $R^a_{0hd} = 2\alpha_{0[h}\delta^a_{d]}$,
(4) $R^a_{0h\hat{d}} = -\alpha_0{}^d\ \delta^a_h + 2\alpha_0\ \delta^a_b\ \alpha^{[b}d^{d]}_h - 2\alpha^{[a}\ d^{c]}_h\ \alpha_0\ \delta^d_c$,
(5) $R^a_{bhd} = 2A^a_{bhd}$,
(6) $R^a_{bh\hat{d}} = A^{ad}_{bh} - (a_0)^2\ \delta^a_h\ \delta^d_b + 4\alpha^{[a}\ \delta^{c]}_h\ \alpha_{[c}\ \delta^d_{b]}$,
(7) $R^a_{bh0} = A^a_{bh0} - \alpha_0C_b\delta^a_h$,
(8) $R^a_{b\hat{h}\hat{d}} = 2A^{ahd}_b$,
(9) $R^a_{b\hat{h}0} = A^{ah0}_b + \alpha_0C^a\delta^h_b$,
(10) $R^a_{\hat{b}hd} = 2(\alpha^b{}_{[h}\delta^a_{d]} - \alpha^a{}_{[h}\delta^b_{d]} - (a_0)^2\delta^a_{[h}\delta^b_{d]})$,
(11) $R^a_{\hat{b}h\hat{d}} = -(\alpha^{bd}\delta^a_h - \alpha^{ad}\delta^b_h + 4\alpha^{[a}\delta^{b]}_c\alpha^{[c}\delta^{d]}_h)$,

(12) $R^a_{\hat{b}h0} = -\alpha^{b0}\delta^a_h + \alpha^{a0}\delta^b_h + 2\alpha^{[a}\delta^{b]}_h\alpha_0 + \alpha_0 C^a\delta^b_h - \alpha_0 C^b\delta^a_h$,
and all other components are zeros or they can be found by the symmetric properties or conjugate of the above components.

Theorem 5. *On G-structure adjoined space, the Ricci tensor components for locally conformal $C_{12}-$ manifold are given by:*

(1) $r_{00} = C^a{}_a + C_a{}^a - 2C^a\, C_a - 2n(\alpha_{00} + (\alpha_0)^2) + (n-1)\alpha^d\, C_d + (n-1)\alpha_d\, C^d$,
(2) $r_{a0} = A^b_{ab0} - (2n-1)\alpha_0\, C_a - (n-1)\alpha_{a0} - (n-1)\alpha_0\, \alpha_a$,
(3) $r_{ab} = C_{ab} - C_a\, C_b - 2\, A^c_{abc} - 2(n-1)\alpha_{ab}$,
(4) $r_{\hat{a}b} = C^a{}_b - C^a\, C_b + (-\alpha_{00} - 2n(\alpha_0)^2 + \alpha^d\, C_d - \alpha^c{}_c - (n-1)\alpha^h\, \alpha_h)\, \delta^a_b$
$+ (n-1)\alpha^a\, \alpha_b - (n-2)\alpha^a{}_b + A^{ac}_{cb} - \alpha^a\, C_b,$

and the remaining components are determined by symmetry, or can be found by using the conjugates of the given components.

Theorem 6. *The locally conformal $C_{12}-$ manifold is an $\eta-$Einstein manifold if and only if, the following conditions hold:*
1) $\lambda = C^a{}_a + C_a{}^a - 2C^a\, C_a - 2n(\alpha_{00} + (\alpha_0)^2) + (n-1)\alpha^d\, C_d + (n-1)\alpha_d\, C^d - \mu,$
2) $A^b_{ab0} = (2n-1)\alpha_0\, C_a + (n-1)\alpha_{a0} + (n-1)\alpha_0\, \alpha_a,$
3) $A^c_{abc} = \frac{1}{2}(C_{ab} - C_a\, C_b) - (n-1)\alpha_{ab},$
4) $A^{ac}_{cb} = (C^a{}_a + C_a{}^a - 2C^a\, C_a - (2n-1)\alpha_{00} + (n-2)\alpha^d\, C_d + (n-1)\alpha_d\, C^d + \alpha^c{}_c$
$+(n-1)\alpha^h\, \alpha_h - \mu)\delta^a_b - C^a{}_b + C^a\, C_b + \alpha^a\, C_b + (n-2)\alpha^a{}_b - (n-1)\alpha^a\, \alpha_b.$

Definition 7. [2] An almost contact metric manifold $(M^{2n+1}, g, \xi, \eta, \Phi)$ is called a manifold of class CR_1 if the curvature tensor R satisfies:

$$g(R(\Phi^2X, \Phi^2Y)\Phi Z, \Phi W) = g(R(\Phi X, \Phi Y)\Phi Z, \Phi W), \quad \forall X, Y, Z, W \in X(M).$$

Whereas, M is called a manifold of class CR_2 if the curvature tensor R satisfies:

$$\begin{aligned} g(R(\Phi^2X, \Phi^2Y)\Phi^2Z, \Phi^2W) &= g(R(\Phi X, \Phi Y)\Phi^2Z, \Phi^2W) + g(R(\Phi X, \Phi^2Y)\Phi Z, \Phi^2W) \\ &\quad + g(R(\Phi X, \Phi^2Y)\Phi^2Z, \Phi W), \quad \forall X, Y, Z, W \in X(M). \end{aligned}$$

On the other hand, M is called a manifold of class CR_3 if the curvature tensor R satisfies:

$$g(R(\Phi^2X, \Phi^2Y)\Phi^2Z, \Phi^2W) = g(R(\Phi X, \Phi Y)\Phi Z, \Phi W), \quad \forall X, Y, Z, W \in X(M).$$

Proposition 8. *A locally conformal $C_{12}-$manifold M is CR_3- manifold if and only if $A^a_{bcd} = 0$ on the G-structure adjoined space.*

Proposition 9. *A locally conformal $C_{12}-$ manifold M is CR_2- manifold if and only if $A^a_{bcd} = 0$ on the G-structure adjoined space.*

Corollary 10. *A locally conformal $C_{12}-$manifold M is CR_2- manifold if and only if M is CR_3- manifold. That is, on M, $CR_2 = CR_3$.*

Proposition 11. *A locally conformal $C_{12}-$manifold M is CR_1- manifold if and only if $A^a_{bcd} = 0$ and $\alpha^a{}_c = -\frac{1}{2}(\alpha_0)^2\,\delta^a_c$ on the G-structure adjoined space.*

Theorem 12. *Let M be a locally conformal $C_{12}-$manifold of class CR_3. Then M is $\eta-$Einstein manifold if and only if, the following conditions satisfied:*
1) $\lambda = C^a{}_a + C_a{}^a - 2C^a\, C_a - 2n(\alpha_{00} + (\alpha_0)^2) + (n-1)\alpha^d\, C_d + (n-1)\alpha_d\, C^d - \mu$,

2) $A^b_{ab0} = (2n-1)\alpha_0\ C_a + (n-1)\alpha_{a0} + (n-1)\alpha_0\ \alpha_a$,
3) $\alpha_{ab} = \frac{1}{2(n-1)}(C_{ab}\ - C_a\ C_b)$; $n > 1$ *and* $\alpha_{[ab]} = 0$,
4) $A^{ac}_{cb} = (C^a{}_a + C_a{}^a - 2C^a\ C_a - (2n-1)\alpha_{00} + (n-2)\alpha^d\ C_d + (n-1)\alpha_d\ C^d + \alpha^c{}_c$
$+(n-1)\alpha^h\ \alpha_h - \mu)\delta^a_b - C^a{}_b + C^a\ C_b + \alpha^a\ C_b + (n-2)\alpha^a{}_b - (n-1)\alpha^a\ \alpha_b$

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On a three-point extension of Chatterjea-type mappings

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In 1972, S. Chatterjea [1] proved a fixed point theorem for mappings that may be discontinuous on their domain: let $T\colon X \to X$ be a mapping on a complete metric space (X, d) with

$$d(Tx, Ty) \leqslant \gamma(d(x, Ty) + d(y, Tx))$$

where $0 \leqslant \gamma < \frac{1}{2}$ and $x, y \in X$. Then T has a unique fixed point. In [2], a class of generalized Chatterjea-type mappings with contraction coefficient $\lambda \in [0, \frac{1}{3})$ was introduced, and a fixed-point theorem for these mappings was proved. A little later C. M. Păcurar and O. Popescu [3] improved this result, see Theorem 2, extending the value of λ to $[0, \frac{1}{2})$.

Definition 1. Let (X, d) be a metric space with $|X| \geqslant 3$. We shall say that $T\colon X \to X$ is a *generalized Chatterjea type mapping* on X if there exists $\gamma \in [0, \frac{1}{2})$ such that the inequality

$$\begin{aligned} &d(Tx, Ty) + d(Ty, Tz) + d(Tx, Tz) \\ &\leqslant \gamma(d(x, Ty) + d(x, Tz) + d(y, Tx) + d(y, Tz) + d(z, Tx) + d(z, Ty)) \end{aligned}$$

holds for all three pairwise distinct points $x, y, z \in X$.

Independently, it was shown in [2] and [3] that usual Chatterjea-type mappings are generalized Chatterjea-type mappings.

Let T be a mapping on the metric space X. A point $x \in X$ is called a *periodic point of period* n if $T^n(x) = x$. The least positive integer n for which $T^n(x) = x$ is called the prime period of x. In particular, the point x is of prime period 2 if $T^2(x) = x$ and $Tx \neq x$.

Theorem 2. *Let (X, d), $|X| \geqslant 3$, be a complete metric space and let the mapping $T\colon X \to X$ satisfy the following two conditions:*

(i) T does not possess periodic points of prime period 2.
(ii) T is a generalized Chatterjea type mapping on X.

Then T has a fixed point. The number of fixed points is at most two.

The following results were obtained by E. Petrov in co-authorship with R. K. Bisht in [4].

Recall that for a given metric space X, a point $x \in X$ is said to be an *accumulation point* of X if every open ball centered at x contains infinitely many points of X.

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Proposition 3. *Let (X,d) be a metric space and let $T\colon X \to X$ be a generalized Chatterjea type mapping. If x is an accumulation point of X and T is continuous at x, then the inequality*

$$d(Tx,Ty) \leqslant \frac{\gamma}{1-\gamma}\left(2d(x,Ty)+d(y,Tx)\right)$$

holds for all points $y \in X$.

Corollary 4. *Let (X,d) be a metric space, $T\colon X \to X$ be a continuous generalized Chatterjea type mapping with $\gamma \in [0,\frac{1}{4})$ and let all points of X be accumulation points. Then T is a Chatterjea type mapping.*

The mappings contracting perimeters of triangles and generalized Kannan type mappings were considered in [5] and [6], respectively. The following three propositions establish connections between these mappings and generalized Chatterjea type mappings.

Proposition 5. *Let (X,d) be a metric space with $|X| \geq 3$, and $T\colon X \to X$ be a mapping contracting perimeters of triangles with a constant $0 \leqslant \alpha < \frac{1}{3}$. Then T is a generalized Chatterjea type mapping.*

Proposition 6. *Let (X,d) be a metric space with $|X| \geq 3$, and $T\colon X \to X$ be a generalized Kannan mapping with a constant $0 \leqslant \lambda < \frac{1}{2}$. Then T is a generalized Chatterjea type mapping.*

Proposition 7. *Let (X,d) be a metric space with $|X| \geq 3$, and $T\colon X \to X$ be a generalized Chatterjea type mapping with a constant $0 \leqslant \gamma < \frac{1}{5}$. Then T is a generalized Kannan mapping.*

Proposition 8. *Generalized Chatterjea type mappings are continuous at fixed points.*

In the following two theorems, we exclude the completeness requirement of the metric space, compare with Theorem 2.

Theorem 9. *Let (X,d), $|X| \geqslant 3$, be a metric space and let the mapping $T\colon X \to X$ satisfy the following four conditions:*

- *(i) T does not possess periodic points of prime period 2.*
- *(ii) T is a generalized Chatterjea type mapping on X.*
- *(iii) T is continuous at $x^* \in X$.*
- *(iv) There exists a point $x_0 \in X$ such that the sequence of iterates $x_n = Tx_{n-1}$, $n = 1,2,\ldots$, has a subsequence x_{n_k}, converging to x^*.*

Then x^* is a fixed point of T. The number of fixed points is at most two.

Theorem 10. *Let (X,d), $|X| \geqslant 3$, be a metric space and let the mapping $T\colon X \to X$ be continuous. Suppose that*

- *(i) T does not possess periodic points of prime period 2.*
- *(ii) T is a generalized Chatterjea type mapping on (M,d), where M is an everywhere dense subset of X.*
- *(iii) There exists a point $x_0 \in X$ such that the sequence of iterates $x_n = Tx_{n-1}$, $n = 1,2,\ldots$, has a subsequence x_{n_k}, converging to x^*.*

Then x^* is a fixed point of T. The number of fixed points is at most two.

Acknowledgments. E. Petrov was supported by the National Research Foundation of Ukraine, project No. 2025.07/0369 "Qualitative methods of nonlinear analysis of heterogeneous structures".

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Introducing the Topological Stability Index

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We introduce the *Topological Stability Index* (TSI), a variance-based scalar measure for persistence barcodes that quantifies the dispersion of persistence lifetimes. Unlike persistent entropy, which depends only on normalized weights, the TSI captures absolute variability and is sensitive to heterogeneous feature scales. We establish fundamental properties of the TSI, including its scaling behavior, invariance under lifetime translation and explicit update formulas under insertion and deletion of bars. We also consider a complementary first-moment-type quantity, the *Topological Signal Index* (TSigI), which captures the typical scale of persistence lifetimes and provides additional interpretability alongside the TSI. We further introduce a normalized version, *cv*TSI, which is scale invariant and admits an explicit algebraic relation to the Rényi entropy of order two. In particular, *cv*TSI is an affine function of the collision probability $\sum_i p_i^2$, and therefore a monotone reparametrization of the Rényi entropy, providing a direct link between variance-based and entropy-based summaries in topological data analysis. Numerical experiments on synthetic data and stochastic time series demonstrate that the TSI captures structural variability complementary to entropy: it is relatively insensitive to deterministic trends, while responding strongly to stochastic fluctuations and variations in persistence magnitude.

Fractional harmonic measure as a tool in minimum Riesz energy problems with external fields

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For the Riesz kernel $\kappa_\alpha(x,y) := |x-y|^{\alpha-n}$ on $\mathbb{R}^n$, where $n \geqslant 2$, $\alpha \in (0,2]$, and $\alpha < n$, we consider the problem of minimizing the Gauss functional

$$\int \kappa_\alpha(x,y)\, d(\mu\otimes\mu)(x,y) + 2\int f_{q,z}\, d\mu, \quad \text{where } f_{q,z} := -q\int \kappa_\alpha(\cdot,y)\, d\varepsilon_z(y),$$

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q being a positive number, ε_z the unit Dirac measure at a fixed point $z \in \mathbb{R}^n$, and μ ranging over all probability measures of finite energy, concentrated on a *quasiclosed* set $A \subset \mathbb{R}^n$ (i.e. such A that can be approximated in outer Riesz capacity $c^*(\cdot)$ by closed sets, cf. [1]). For any $z \in A^u \cup (\mathbb{R}^n \setminus \mathrm{Cl}_{\mathbb{R}^n} A)$, where A^u is the set of all inner α-ultrairregular points for A, we provide necessary and sufficient conditions for the existence of the minimizer $\lambda_{A,f_{q,z}}$, establish its alternative characterizations, and describe its support, thereby discovering new interesting phenomena. In detail, $z \in \partial_{\mathbb{R}^n} A$ is said to be *inner α-ultrairregular* for A if the inner α-harmonic measure ε_z^A [3] is of finite energy, or alternatively if the inverse A_z^* of A with respect to the unit sphere centered at z is of finite inner capacity $c_*(A_z^*)$. We show that for any $z \in A^u \cup (\mathbb{R}^n \setminus \mathrm{Cl}_{\mathbb{R}^n} A)$, $\lambda_{A,f_{q,z}}$ exists if and only if either $c_*(A) < \infty$, or $q \geqslant H_z$, where $H_z := 1/\varepsilon_z^A(\mathbb{R}^n) \in [1, \infty)$. Thus, for any closed A, any $z \in A^u$, and any $q \geqslant H_z$ — even arbitrarily large, no compensation effect occurs between the two oppositely signed charges, $-q\varepsilon_z$ and $\lambda_{A,f_{q,z}}$, carried by the same conductor A, which at the first glance seems to contradict our physical intuition. Another interesting phenomenon is that, if A is closed and of unbounded boundary, then for any $z \in A^u \cup (\mathbb{R}^n \setminus A)$, the support of $\lambda_{A,f_{H_z,z}}$ is noncompact, whereas that of $\lambda_{A,f_{H_z+\beta,z}}$ is already compact for any $\beta \in (0, \infty)$ — even arbitrarily small. The above-mentioned results were published in the author's recent paper [5], and they substantially improve some of the latest ones on the problem in question, e.g. those in [2, 4].

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On loops that satisfy $x \cdot (x \cdot yx)z = (x \cdot xy) \cdot xz$

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An LTWC is a loop that satisfies $x \cdot (x \cdot yx)z = (x \cdot xy) \cdot xz$. LTWC loops are proved to be power associative and left conjugacy closed (LCC). An LCC loop is LTWC if and only if $x(x \cdot yx) = (x \cdot xy)x$. Connections to left Bol loops, left Cheban loops and loops satisfying $(xy \cdot x) \cdot xz = x \cdot (yx \cdot x)z$ (LWPC) are also considered.

This paper is mainly concerned with loops fulfilling the law

$$x \cdot (x \cdot yx)z = (x \cdot xy) \cdot xz \qquad \text{(LTWC)}$$

This law was introduced in [2], where it is dubbed (Q_{12}). It may be regarded as a 'twist' of the law

$$(xy \cdot x) \cdot xz = x \cdot (yx \cdot x)z \qquad \text{(LWPC)}$$

that was first formulated by Phillips in [4] and investigated in [1, 3].

$$y(xz \cdot x) \cdot x = yx \cdot (zx \cdot x) \qquad \text{(RTWC)}$$
$$zx \cdot (x \cdot yx) = z(x \cdot xy) \cdot x \qquad \text{(RWPC)}$$
$$yx \cdot xz = (y \cdot zx)x \qquad \text{(RCh)}$$

Lemma 1. *Let x and y be elements of a loop Q.*
(i) If Q fulfills (LTWC) or (RWPC), then $x(x \cdot yx) = (x \cdot xy)x$, i.e. $L_x^2 R_x = R_x L_x^2$.
(ii) If Q fulfills (RTWC) or (LWPC), then $(xy \cdot x)x = x(yx \cdot x)$, i.e. $R_x^2 L_x = L_x R_x^2$.
(iii) If Q is left or right Cheban, then $x \cdot xy = yx \cdot x$, i.e. $L_x^2 = R_x^2$.

Theorem 2. *Let Q be a loop. Then*

(i): *Q fulfills (LTWC) if and only if Q is an LCC loop in which $x(x \cdot yx) = (x \cdot xy)x$ for all $x, y \in Q$; and*

(ii): *Q fulfills (LWPC) if and only if Q is an LCC loop in which $x(yx \cdot x) = (xy \cdot x)x$ for all $x, y \in Q$.*

Theorem 3. *Every LTWC loop is power associative.*

Theorem 4.

(1) If an LWPC-loop is a left Bol loop, then it is an LTWC-loop.
(2) A left Bol loop is an LTWC-loop if and only if it is a left Burn loop.

НТБ ОНТУ

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Invariant Geometric Properties in Bubble Contraction During Fluid Infiltration Between Parallel Plates

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The property of σ-superstarlikeness is invariant in the interior version of the Hele–Shaw flow problem for

$$\sigma \in \left[\frac{1}{2}, +\infty\right) \cup \bigcup_{n=1}^{\infty} \left[\frac{1}{4n+2}, \frac{1}{4n}\right].$$

This result was recently established by Aron [1, 2]. In contrast, the invariance of starlikeness and strongly starlikeness (that is, σ-superstarlikeness for $\sigma \geq 1$) in the exterior Hele–Shaw problem was incorrectly claimed. We present a general geometric property, more general than the notion of starlikeness, which remains invariant in the exterior version of the Hele–Shaw flow problem. Specifically, the property of σ-superstarlikeness for

$$\sigma \in \bigcup_{n=1}^{\infty} \left[\frac{1}{4n}, \frac{1}{4n-2}\right]$$

is shown to be invariant in the exterior version of the Hele–Shaw problem.

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НТБ ОНТУ

Telescoping transformations for reciprocal series of second-order recurrence sequences

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The aim of this paper is to develop a unified approach based on a discrete Abel-type summation formula (summation by parts). The main idea is to transform reciprocal series containing three or more factors in the denominator into expressions exhibiting a partial telescoping or alternating structure. This transformation reduces the complexity of the sums and makes it possible to derive families of identities in a uniform way. Our results may be viewed as a continuation and extension of the results obtained in [1], where several three-parameter families of finite reciprocal sums involving Horadam numbers were evaluated.

We consider sequences $w_n = w_n(a,b;p)$ defined by the recurrence relation $w_n = pw_{n-1} + w_{n-2}$, $n \geq 2$, with initial values $w_0 = a$ and $w_1 = b$. This class includes, as special cases, the Fibonacci sequence $F_n = w_n(0,1;1)$ and the Lucas sequence $L_n = w_n(2,1;1)$, as well as several related sequences. We also use the associated sequences $u_n = w_n(0,1;p)$ and $v_n = w_n(2,p;p)$.

Theorem 1. *Let r, s, and l be nonnegative integers. If m is odd, then*

$$\sum_{k=1}^{\infty} \frac{1}{w_{mk+s} w_{m(k+1)+s} \prod_{i=0}^{n-3} w_{r(k+i)+l}} = \frac{1}{u_m w_{m+s}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\big(w_{rk+l} + w_{r(k+n-2)+l}\big) u_{mk}}{w_{m(k+1)+s} \prod_{i=0}^{n-2} w_{r(k+i)+l}}.$$

If m is even, then

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{w_{mk+s} w_{m(k+1)+s} \prod_{i=0}^{n-3} w_{r(k+i)+l}} = \frac{1}{u_m w_{m+s}} \sum_{k=1}^{\infty} (-1)^{k} \frac{\big(w_{rk+l} + w_{r(k+n-2)+l}\big) u_{mk}}{w_{m(k+1)+s} \prod_{i=0}^{n-2} w_{r(k+i)+l}}.$$

By selecting appropriate values of r, m and l, we obtain from Theorem 1 the following classical-type and alternating identities involving Fibonacci and Lucas numbers:

$$\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+1}F_{k+2}}=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{F_{k+1}},\qquad \sum_{k=1}^{\infty}\frac{(-1)^k}{F_kF_{2k}F_{2k+2}}=\sum_{k=1}^{\infty}(-1)^k\frac{F_{k+2}L_k}{F_{k+1}^2L_{k+1}},$$
$$\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+1}F_{k+2}F_{k+3}}=\frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{F_{k+1}+2F_k}{F_{k+1}F_{k+2}F_{k+3}},$$
$$\sum_{k=1}^{\infty}\frac{(-1)^k}{F_kF_{k+1}F_{2k}F_{2k+2}}=\sum_{k=1}^{\infty}(-1)^k\frac{F_{k+1}+2F_k}{F_{k+1}F_{k+2}F_{2k+2}}L_k.$$

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On n-convex $(2n-2)$-dimensional submanifolds in E^{2n}

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Let us recall the following definition (see [1, §16]).

Definition 1. A subset C of the Euclidean space E^n is called k-convex if through each point $x \in E^n \setminus C$ there passes a k-dimensional plane π_x that does not intersect C.

The following theorem is an n - dimensional generalization of our recent result obtained in [2].

Theorem 2. *If $S \subset E^{2n}$ is an n-convex closed $(2n-2)$-dimensional submanifold which is C^2-embedded into E^{2n}, then it admits a mapping of degree one to $S^{n-1}\times S^{n-1}$. If S is non-orientable, we assume that the degree is taken modulo 2.*

Corollary 3. *Neither S^{2n-2} nor $S^k \times S^{2n-k-2}$, $1 \le k \le n-2$, admits a C^2-embedding into E^{2n} as a n-convex submanifold.*

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НТБ ОНТУ

On partial preliminary group classification of some class of (1+3)-dimensional Monge-Ampere equations. Three-dimensional Lie algebras

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The study of differential geometry, Kähler geometry and the Calabi conjecture, theoretical physics, optimal mass transportation, reflector and lens design in geometric optics, semi-geostrophic equations in meteorological modeling, elasticity, image processing and image registration, and optimal control in mathematical finance all lead, in various dimensions, to equations of Monge-Ampère type — reflecting their remarkable universality across mathematics and its applications. Numerous papers and monographs have been dedicated to the analysis of these classes of equations from various methodological perspectives.

We consider the following class of $(1+3)$-dimensional Monge-Ampère equations:

$$\det\left(u_{\mu\nu}\right) = F\left(x_0, x_1, x_2, x_3, u, u_0, u_1, u_2, u_3\right),$$

where $u = u(x)$, $x = (x_0, x_1, x_2, x_3) \in M(1,3)$, $u_{\mu\nu} \equiv \dfrac{\partial^2 u}{\partial x_\mu \partial x_\nu}$, $u_\alpha \equiv \dfrac{\partial u}{\partial x_\alpha}$, $\mu, \nu, \alpha = 0, 1, 2, 3.$

Here $M(1,3)$ is a four-dimensional Minkowski space, F is an arbitrary real smooth function.

We continue using the classical Lie-Ovsiannikov approach for partial preliminary group classification of this class. In [27, 28] we have performed the classification using one and two-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$.

I plan to present some of the results obtained using three-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$.

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Mixed volume inequalities for zonoids and log-submodularity of volume

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НТБ ОНТУ

1. Zonoids and log-submodularity. A zonotope is the Minkowski sum of line segments – a linear projection of a cube. A zonoid is a Hausdorff limit of zonotopes. For any zonoid A and zonoids $B_1, \dots, B_k$ in $\mathbb{R}^n$, Fradelizi, Madiman, Meyer and Zvavitch conjectured the log-submodularity inequality

$$\operatorname{vol}(A) \cdot \operatorname{vol}(A + B_1 + \cdots + B_k) \ \le\ \prod_{i=1}^{k} \operatorname{vol}(A + B_i),$$

which expresses a multiplicative diminishing-returns principle for Minkowski addition. This property is distinct from the classical log-concavity of volume, which concerns Minkowski combinations rather than sums. The authors have proven the conjecture in dimension three in [1].

The *mixed volume* $V(K_1, \dots, K_n)$ is defined by the polarization identity

$$V(K_1, \dots, K_n) = \frac{1}{n!} \sum_{j=1}^{n} (-1)^{n+j} \sum_{1 \le i_1 < \cdots < i_j \le n} \operatorname{vol}(K_{i_1} + \cdots + K_{i_j}),$$

and satisfies $V(K, \dots, K) = \operatorname{vol}(K)$.

Using mixed volumes, the conjecture is equivalent to a family of inequalities that we call *Bezout-type inequalities*:

$$\operatorname{vol}(A)^{k-1}\, V(A[n-k], B_1, \dots, B_k) \ \le\ \frac{n^k (n-k)!}{n!} \prod_{i=1}^{k} V(A[n-1], B_i), \qquad 1 \le k \le n.$$

The name reflects a structural analogy with Bézout's theorem in algebraic geometry: just as the intersection number of hypersurfaces is bounded by the product of their degrees, here the mixed volume (measuring "intersection" of Minkowski sums) is bounded by a product of lower-order mixed volumes. This connection suggests a deep interplay between convex geometry and intersection theory.

2. Reduction to a hypercube inequality. For arbitrary dimension n, the Bezout-type inequality reduces to a purely geometric statement about the $(n-1)$-dimensional cube $[0,1]^{n-1}$. Assign non-negative weights s_p to the 2^{n-1} vertices. The inequality becomes

$$\Big(\sum_{p \in \{0,1\}^{n-1}} s_p \Big)^{n-2} \Big(\sum_{\substack{S \subset \{0,1\}^{n-1} \\ |S| = n}} \big(\prod_{p \in S} s_p\big) c_S \Big) \ \le \prod_{\text{facets } F \text{ of } [0,1]^{n-1}} \Big(\sum_{p \in F \cap \{0,1\}^{n-1}} s_p \Big),$$

where c_S denotes the normalized $(n-1)$-dimensional volume of the simplex with vertex set S. Thus the hypercube inequality compares a weighted total volume of all $(n-1)$-dimensional simplices inscribed in the cube with the product of the total weights on each facet.

3. Main theorem.

Theorem 1 (Hypercube inequality for $n = 4$)**.** *Let $s_1, \ldots, s_8 \geq 0$ be vertex weights. Then*

$$\Big(\sum_{i=1}^{8} s_i\Big)^2 \Bigg[\Big(\sum_{i=1}^{4} s_i\Big)\Big(\sum_{5\leq i<j<k\leq 8} s_i s_j s_k\Big) + \Big(\sum_{i=5}^{8} s_i\Big)\Big(\sum_{1\leq i<j<k\leq 4} s_i s_j s_k\Big)$$
$$+ (s_1+s_2)(s_3+s_4)(s_5s_6+s_7s_8) + (s_1+s_3)(s_2+s_4)(s_5s_7+s_6s_8)$$
$$+ (s_1+s_4)(s_2+s_3)(s_6s_7+s_5s_8) + 2s_1s_2s_6s_7 + 2s_2s_3s_5s_8\Bigg]$$
$$\leq \Big(\sum_{i=1}^{4} s_i\Big)\Big(\sum_{i=5}^{8} s_i\Big)\Big(\sum_{i\in\{1,3,5,7\}} s_i\Big)\Big(\sum_{i\in\{2,4,6,8\}} s_i\Big)\Big(\sum_{i\in\{1,2,5,6\}} s_i\Big)\Big(\sum_{i\in\{3,4,7,8\}} s_i\Big).$$

Equality holds iff at least one of the six face sums is zero.

4. Geometric consequences. The theorem directly implies the Bezout-type inequality for $n = 4$, hence the log-submodularity conjecture for zonoids in $\mathbb{R}^4$. Moreover, it confirms the projection inequality

$$\mathrm{vol}(A)^{k-1}\,\mathrm{vol}(P_{[b_1,\ldots,b_k]^\perp}A) \;\leq\; \prod_{i=1}^{k}\mathrm{vol}(P_{b_i^\perp}A)$$

for any orthonormal basis $\{b_1, \ldots, b_4\}$ and any zonoid $A \subset \mathbb{R}^4$. Geometrically, this bounds the volume of a zonoid projected onto a k-codimensional subspace by the product of its projections onto k coordinate hyperplanes. The equality case occurs when at least one of those projections has zero volume – a natural geometric degeneracy.

The algebraic certificate derived from the cube geometry suggests a systematic method for higher dimensions. The hypercube polynomial is invariant under the cube's symmetry group, and its Newton polytope has a rich combinatorial structure. The analogy with Bézout's theorem opens a promising link to intersection theory and toric geometry, which may guide the search for certificates in higher dimensions.

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On one-dimensional foliations on the plane

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Definition 1. A mapping $p \in C(X,Y)$ is called *compact-covering* if for every compact subset $L \subset Y$ there exists a compact subset $K \subset X$ such that $p(K) = L$.

Theorem 2 (See [1]). *Let Δ be a (topological) one-dimensional foliation on the plane $\mathbb{R}^2$. Then the quotient mapping $p : \mathbb{R}^2 \to Y = \mathbb{R}^2/\Delta$ onto its space of leaves Y is compact-covering.*

We will give an exposition of the proof of this theorem, which is slightly different from the one given in [1].

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НТБ ОНТУ

Differentiable structures on line with n+1 origins

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extended the arguments from [1] to obtain a classification of differentiable structure on the line with $n+1$ origins L_n up to a diffeomorphisms which fix Hausdorff-violating points.

Definition 1. Let τ be standard topology on $\mathbb{R}$. Let L_n be a disjoint union of $\mathbb{R}$ with the set $\{0_1, 0_2, \dots, 0_n\}$ endowed with the following topology:

$$\eta = \tau \cup \coprod_{i=1}^{n} \big\{ (W \setminus \{0\}) \cup \{0_i\} : 0 \in W \in \tau \big\}$$

with additional elements being the open neighborhoods of 0 in which 0 is replaced by 0_i.

The space L_n is locally Euclidean and T_1, but not T_2 as the set $\mathrm{Hv} := \{0, 0_1, 0_2, \dots, 0_n\}$ consists precisely of points for which the T_2 separation axiom fails to hold. Such points, are often called *Hausdorff-violating.*

Definition 2. Let A be a differentiable atlas on L_n. If A consists of charts of the form $\{(U_i, \phi_i)\}_{i=0}^{n}$ where $U_i := \mathbb{R} \setminus \{0\} \cup \{0_i\}$ ($U_0 := \mathbb{R}$), $\phi_i(0_i) = 0$ ($\phi_0(0) = 0$) and ϕ_i maps $(0; +\infty) \to (0; +\infty)$, then A is called the *minimal atlas* on L_n.

For any two charts in miminal atlas the transition map is a monotone $\mathcal{C}^r$-diffeomorphism of $\mathbb{R} \setminus \{0\}$. Therefore there exists a unique *continuous extension* of $\phi_j \phi_i^{-1}$ to the homeomorphism of $\mathbb{R}$ such that $0 \mapsto 0$. Let $T_{i,j}^{(\mathsf{A})}$ ($T_j^{(\mathsf{A})}$ if $i = 0$) denote such extension in some atlas A .

In general, if f is a $\mathcal{C}^r$-diffeomorphism which acts on Hv by a permutation σ, then we have the following commutative diagrams and corresponding relations between extensions of transition maps.

$$\begin{tikzcd}
& U_j \arrow[r, "f"] \arrow[dl, "\phi_j"] & V_q \arrow[dr, "\psi_q"] & \\
\mathbb{R} \arrow[rrr, "f_{j,q}", "\psi_j f \phi_q^{-1}"'] \arrow[dd, shift left, "T^{(\mathsf{A})}_{j,i}"] & & & \mathbb{R} \arrow[dd, shift right, "T^{(\mathsf{B})}_{q,p}"'] \\
& U_i \arrow[r, "f"] \arrow[dl, "\phi_i"] & V_p \arrow[dr, "\psi_p"] & \\
\mathbb{R} \arrow[rrr, "f_{i,p}", "\psi_i f \phi_p^{-1}"'] \arrow[uu, shift left, "T^{(\mathsf{A})}_{i,j}"] & & & \mathbb{R} \arrow[uu, shift right, "T^{(\mathsf{B})}_{p,q}"']
\end{tikzcd}
\implies
\begin{tikzcd}
\mathbb{R} \arrow[r, "f_{0,\sigma(0)}", "\psi_{\sigma(0)} f \phi_0^{-1}"'] \arrow[d, "T^{(\mathsf{A})}_{0,i}"] & \mathbb{R} \arrow[d, "T^{(\mathsf{B})}_{\sigma(0),\sigma(i)}"] \\
\mathbb{R} \arrow[r, "f_{i,\sigma(i)}", "\psi_{\sigma(i)} f \phi_i^{-1}"'] & \mathbb{R}
\end{tikzcd}$$

$$\begin{cases} f_{i,p} = T^{(\mathsf{B})}_{q,p} f_{j,q} T^{(\mathsf{A})}_{i,j} \\ T^{(\mathsf{B})}_{q,p} = f_{i,p} T^{(\mathsf{A})}_{j,i} f^{-1}_{j,q} \end{cases} \qquad T^{(\mathsf{B})}_{\sigma(0),\sigma(i)} = f_{i,\sigma(i)} T^{(\mathsf{A})}_{0,i} f^{-1}_{0,\sigma(0)}$$

For each $r \in \mathbb{N} \cup \{\infty\}$ let

- $\mathcal{W}_{\mathbb{R}} := \mathcal{W}^r(\mathbb{R}, 0)$ be the group of homeomorphisms of $\mathbb{R}$ such that any element of $\mathcal{W}_{\mathbb{R}}$ maps 0 to itself and is a monotone $\mathcal{C}^r$-diffeomorphism of $\mathbb{R} \setminus \{0\}$;
- $\mathcal{D}_{\mathbb{R}} := \mathcal{D}^r(\mathbb{R}, 0)$ be the group of $\mathcal{C}^r$-diffeomorphisms of $\mathbb{R}$ which fix 0. Note that $\mathcal{D}_{\mathbb{R}}$ is a subgroup of $\mathcal{W}_{\mathbb{R}}$.

Proposition 3. *Let $\mathfrak{A}$ be some differentiable structure on L_n. Then $\mathfrak{A}$ contains a minimal atlas* A *as subatlas. Moreover, there is a one-to-one correspondence between minimal atlases and elements $\Xi^{(\cdot)} := (T_i^{\cdot})_{i=1}^n$ of the direct product $\prod_{i=1}^n \mathcal{W}_{\mathbb{R}}$.*

If f fix Hausdorff-violating points, the corresponding permutation σ is trivial and the theorem follows

Theorem 4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two differentiable structures (maximal atlases) on L_n, then the following conditions are equivalent:*

(i) *There exists a diffeomorphism $(L_n, \mathfrak{A}) \to (L_n, \mathfrak{B})$ which fix all Hausdorff-violating points;*
(ii) *For any pair of minimal atlases $\mathsf{A} \in \mathfrak{A}$ and $\mathsf{B} \in \mathfrak{B}$, elements $\Xi^{(\mathsf{A})}$ and $\Xi^{(\mathsf{B})}$ lie in a same orbit of action*

$$\mathcal{D}_{\mathbb{R}} \times \prod_{i=1}^{n} \mathcal{D}_{\mathbb{R}} \times \prod_{i=1}^{n} \mathcal{W}_{\mathbb{R}} \to \prod_{i=1}^{n} \mathcal{W}_{\mathbb{R}}$$
$$(f_0, f_1, \ldots, f_n).(T_1, \ldots, T_n) \mapsto (f_1 T_1 f_0^{-1}, \ldots, f_n T_n f_0^{-1})$$

The definition of $\mathcal{C}^r$-structure, $\mathcal{C}^r$-atlas and $\mathcal{C}^r$-diffeomorphism can be found in the book by J. Lee [3] and paper of F. Takens [2] with references therein.

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One-Dimensional Non-Hausdorff Manifolds and CW Complexes

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We study one-dimensional non-Hausdorff manifolds that are "similar" to graphs with split vertices. It is shown that if M is a connected one-dimensional non-Hausdorff manifold such that the set of its "non-Hausdorff" points is locally finite, and each component of its complement has a countable base, then there exists a quotient map $\pi\colon M \to \Gamma$ onto an open one-dimensional CW complex, which maps the non-Hausdorff points of M to the vertices of Γ.

Moreover, Γ is the minimal Hausdorff quotient of M, that is, for every continuous map $f\colon M \to N$ into a Hausdorff space N, there exists a unique continuous map $\hat{f}\colon \Gamma \to N$ such that $f = \hat{f} \circ \pi$.

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Emden–Fowler Type Equations in the Theory of Geodesic Mappings

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By a generalized Emden–Fowler equation one usually means an equation of the form

$$y'' + f(x)y^n = 0.$$

A considerable number of scientific works have been devoted to the study of this equation and related nonlinear differential models. In particular, significant contributions to the development of this field were made by researchers of Odesa I. I. Mechnikov National University, including V. M. Evtukhov, A. A. Stekhun, M. O. Bilozerova, and other scholars [1, 2].

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The geometric properties of solutions to this equation have also been investigated in connection with applications in the theory of relativity. In particular, for $n=2$, one obtains the Kustaanheimo–Qvist solution, which represents one of the solutions to the Einstein field equations:

$$R_{ij} - \frac{R}{2} g_{ij} = E_{ij}.$$

We investigate an Emden–Fowler type equation of the form

$$y'' + pyy' + qy^3 = 0.$$

Problems concerning geodesic mappings of a broad class of pseudo-Riemannian spaces can be reduced to this equation. These include, in particular, Einstein spaces, almost Einstein spaces, spaces whose degree of mobility with respect to geodesic mappings exceeds two, and many other types of special pseudo-Riemannian spaces [3, 4, 5].

A geodesic mapping is defined as a one-to-one correspondence between points of pseudo-Riemannian spaces under which geodesic lines of one space are mapped onto geodesic lines of another space. If such a mapping is not homothetic, it is called nontrivial.

The study of the properties of Emden–Fowler type equations has made it possible to establish that complete pseudo-Riemannian spaces for which this equation admits solutions do not allow nontrivial geodesic mappings.

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Invariant idempotent $*$-measures generated by generalized iterated function systems

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Iterated Function Systems (IFSs), introduced by John Hutchinson [1], provide a powerful tool for constructing self-similar fractal objects, including sets, probability measures, and their generalizations. The concept of IFS was later extended to generalized iterated function systems (GIFSs). In this setting, for a metric space X, maps of the form $X^m \to X$ are considered. Such systems were introduced by Alexandru Mihail and Radu Miculescu [2]. They defined the notion of an invariant element in the hyperspace $\exp(X)$ associated with these systems and proved its existence and uniqueness under natural contractivity conditions.

A GIFS on X of order m is a finite set of maps $g_i\colon X^m \to X$, $i = 1, \dots, n$, where g_i is a *generalized Matkowski contraction of degree* m. The last means that there exists nondecreasing $\varphi\colon [0, \infty) \to [0, \infty)$ such that $\varphi^{(k)}(t) \to 0$ as $k \to \infty$, and

$$d(g_i(x), g_i(y)) \le \varphi(d(x, y)), x, y \in X^m,$$

where X^m is endowed with the maximum metric. For results concerning invariant objects appearing from GIFSs, see [3].

In [4], GIFSs acting on the space of probability measures were studied. In particular, it was shown that, under suitable contractivity conditions, given GIFS $\{g_1, g_2, \dots, g_n\}$ of order m and probabilities $p_i \ge 0$ with $\sum\limits_{i=1}^{n} p_i = 1$, there exists an invariant Hutchinson measure, that is, a measure $\mu \in P(X)$ satisfying

$$\mu = \sum_{i=1}^{n} p_i \cdot P(g_i)(\mu \otimes \mu \otimes \cdots \otimes \mu).$$

We extend the results of [5], where the existence of invariant idempotent $*$-measures was established, to the case of GIFSs. In particular, for a given GIFS and a triangular norm $*$, we prove the existence and uniqueness of an invariant $*$-measure. Furthermore, we modify the notion of GIFS by considering maps not only of the form $X^m \to X$, but also maps defined on the G-symmetric power of a space, that is, $SP^m_G(X) \to X$.

Theorem 1. *Let (X, d) be a compact metric space and let $g_i\colon SP^m_G(X) \to X$, $i = 1, \dots, n$, be SP^m_G-generalized Matkowski contractions, where G is a subgroup of the symmetric group S_m. Let $*$ be a triangular norm and $\alpha = \vee_{i=1}^{n} \alpha_i * \delta_i$ be an idempotent $*$-measure on $\{1, 2, \dots, n\}$. Then there exists a unique invariant $*$-measure for the given SP^m_G-GIFS and α.*

We also note that our proof of the existence and uniqueness of invariant idempotent $*$-measures does not rely on metrization of the space of measures and is relatively simple.

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НТБ ОНТУ

Coding of flows with singularities on the boundary of the two-dimensional disk

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We investigate topological properties, possible structures, classifications and coding of flows on a two-dimensional disk, which have a finite number of separatrixes and all singular points of the flow lie on the boundary of the disk. Trees with selected leaves are used to classify such flows. We construct the code for a flow with one singular point on the boundary.

Definition 1. A distingushed graph of a flow is a planar graph obtained from a bivalent graph by deleting the outer vertex and the half-edges incident to it, and equipped with the following additional data:

(1) all edges are oriented;
(2) some leaves of the graph are distinguished;
(3) at each vertex of the graph, a sector is distinguished; this sector can be specified by an ordered pair of adjacent half edges incident to the vertex: the first half-edge corresponds to the eparatrix entering the boundary of the sector, and the second to the separatrix leaving it.

Theorem 2. *Let a flow be given on the 2-disk, having a finite number of singular points and separatrices, and suppose that all singular points lie on the boundary of the 2-disk. Then the distinguished graph of such a flow is a tree.*

Theorem 3. *The flows on the 2-disk that have a finite number of singular points and separatrices, with all singular points lying on the boundary of the 2-disk, are topologically equivalent with preservation of orientation if and only if their distinguished graphs are equivalent.*

Definition 4. A 1-flow on the 2-disk D^2 is a flow that has a finite number of separatrices, exactly one singular point, and this point lies on the boundary of the 2-disk ∂D^2.

Definition 5. The distinguished graph of a 1-flow is a two-colored rooted tree with labels on its edges and with ordering of the lower edges at each vertex.

Two distinguished graphs are called equivalent if there exists an isomorphism between them that preserves the colors, labels, and orderings of the lower edges, and maps the root to the root.

Corollary 6. *Two 1-flows on the 2-disk are topologically trajectory equivalent if and only if their distinguished graphs are equivalent.*

In the article, we introduce the notion of a 1-code, for which the following statements hold:

Theorem 7. *Two* 1*-flows are topologically trajectory equivalent if and only if their codes are identical.*

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НТБ ОНТУ

Some classes of surfaces with Grassmann image of constant curvature in the Minkowski space

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The range of questions of differential geometry of surfaces in multidimensional spaces is significantly expanded if we use the properties of their Grassmann images. This work presents a study of non-isotropic minimal surfaces and surfaces with flat normal connectivity in the Minkowski space ${}^1R_4$ that have a non-degenerate Grassmann image of constant curvature $\overline{K}$.

Let V^2 be a two-dimensional surface of the class $C^k, k \geq 1$ in the Minkowski space ${}^1R_4$ with metric form $ds^2 = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$. A surface in the space ${}^1R_4$ is called *time-like (space-like)* if its tangent plane at every point is time-like (space-like).

The number $H^k = \frac{g^{ij}L_{ij}^k}{2}$ is called the mean curvature of the surface in the direction of the normal vector $\bar{\xi}_k$, and the vector $H = H^1\bar{\xi}_1 + H^2\bar{\xi}_2$ is the mean curvature vector. The surfaces of the Minkowski space with zero mean curvature vector will be called minimal surfaces as in Euclidean space.

Theorem 1. *A time-like minimal surface has a non-degenerate Grassmann image of constant curvature* $\overline{K} = 1$ *if and only if it is a hypersurface of a three-dimensional subspace of the Minkowski space. Time-like minimal surfaces with a non-degenerate Grassmann image of constant curvature* $\overline{K} = 1$ *exist.*

Theorem 2. *In the Minkowski space, there are no time-like minimal surfaces with a non-degenerate Grassmann image of constant curvature other than* 1*.*

Theorem 3. *A space-like minimal surfaces of the Minkowski space with a non-degenerate Grassmann image of the constant curvature* $\overline{K}$ *can only be hypersurfaces of three-dimensional subspaces of the Minkowski space (i.e. can only have curvature* $\overline{K} = -1$*), their Gauss curvature should be non-constant. In the Minkowski space there exist space-like minimal surfaces with a non-degenerate Grassmann image of the constant curvature* $\overline{K}$*.*

We have described [1] all two-dimensional non-isotropic surfaces with flat normal connectivity in the Mincowski space, whose non-degenerate Grassmann image has constant curvature. The following theorem has been proven

Theorem 4. *For every $k \in [0,1]$, in the Minkowski space, there exists a two-dimensional time-like surface V^2 of the class $C^n, n \geq 3$, with flat normal connection the non-degenerate Grassmann image of which has the constant curvature k. For every $k \in (-\infty, -1]$, in the Minkowski space, there exists a two-dimensional space-like surface V^2 of the class $C^n, n \geq 3$, with flat normal connection the non-degenerate space-like Grassmann image that has the constant curvature k. For every $k \in [0, \infty)$, in the Minkowski space, there existst a two-dimensional space-like surface V^2 of the class $C^n, n \geq 3$, with flat normal connection the non-degenerate time-like Grassmann image of which has the constant curvature k.*

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On a total mixed curvature of closed curves

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Given a smooth closed curve γ with nowhere vanishing curvatures $k_1, \dots, k_j$ in $\mathbb{E}^n$, we consider the well-defined integral quantity $J_j(\gamma) = \oint\limits_\gamma \sqrt{k_{j-1}^2 + k_j^2 + k_{j+1}^2}\, ds$, which arises as the total curvature or the total length of certain Frenet frame indicatrices of γ, viewed as smooth closed curves in Grassmann manifolds $G(j,n)$ embedded in $\mathbb{E}^{\binom{n}{j}}$, see [3].

For instance, for $\gamma \subset \mathbb{E}^3$ the quantity J_2, referred to as the total mixed curvature of γ, can be interpreted as the total curvature of the tangent indicatrix, the total curvature of the binormal indicatrix, or the length of the principal normal indicatrix associated with γ, where all three indicatrices are viewed as smooth closed curves in $S^2 \subset \mathbb{E}^3$, see [6], [1].

We address the problem of determining sharp lower bounds for $J_j(\gamma)$. It is known that if j is odd, then $J_j > 2\pi$, and this estimate is sharp [3]. On the other hand, if j is even, then the problem remains open and appears to be challenging even in the case of $j = 2$.

Problem 1. What is the infimum of J_2 over the space Ω of all smooth closed curves in $\mathbb{E}^n$, $n \geq 3$, with nowhere vanishing curvatures k_1 and k_2:

$$\inf_{\gamma \in \Omega} \oint\limits_\gamma \sqrt{k_1^2 + k_2^2 + k_3^2}\, ds \ = ?$$

For curves with nonvanishing constant curvatures in $\mathbb{E}^4$, one has $J_2 \geq 2\sqrt{5}\pi$, see [4]. Furthermore, any circle can be realized as a limit curve for the space of smooth closed curves with nowhere vanishing curvatures k_1 and k_2 in $\mathbb{E}^4$, but the limit value of J_2 cannot be less than $2\sqrt{5}\pi$, see [5]. Thus, it was conjectured that the estimate $J_2 \geq 2\sqrt{5}\pi$ holds for all smooth closed curves with nowhere vanishing curvatures k_1 and k_2 in $\mathbb{E}^4$.

It turns out that the conjectured lower bound for J_2 must be weakened. Namely, the following statements hold true.

Proposition 2. *For an arbitrary smooth closed curve γ with nowhere vanishing curvatures k_1 and k_2 in $\mathbb{E}^n$, $n \geq 3$, there exists a smooth deformation which decreases the value of J_2. Thus, the desired infimum of J_2 can only be attained in the limit by curves that no longer belong to Ω.*

Proposition 3. *For any $\varepsilon > 0$, there exists a smooth closed curve γ with nowhere vanishing curvatures k_1 and k_2 in $\mathbb{E}^n$, $n \geq 3$, such that $J_2(\gamma) < 4\pi + \varepsilon$.*

Proposition 4. *For any $\varepsilon > 0$, there exists a smooth closed curve γ with nowhere vanishing curvatures k_1 and $k_2 \equiv const$ in $\mathbb{E}^n$, $n \geq 3$, such that $J_2(\gamma) < 4\pi + \varepsilon$.*

The corresponding curves in Propositions 3-4 are constructed explicitly and independently.

We also note that Proposition 4, when compared with Proposition 3, illustrates the validity of the h-principle for closed curves with constant torsion in $\mathbb{E}^3$ established recently in [2].

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Menshov—Trokhymchuk Type Theorem in a Finite-Dimensional Banach Algebra.

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We weaken the conditions of monogenity for functions that take values in an arbitrary finite dimensional commutative algebra over the field of complex numbers. The monogenity of the function is understood as a combination of its continuity and the existence of the Gâteaux derivative [2].

The main result generalizes classical Menchov theorem, its part related to K''' property, to the case of functions monogenic in finite dimensional algebras, when functions are defined in specific linear span of the algebra [1].

Consider n-dimentional commutative associative algebra $\mathbb{A}_n^m$ with unit 1 over the field $\mathbb{C}$, where exists a Cartan's basis $\{I_k\}_{k=1}^n$ and norm $\|\lambda_1 I_1 + \lambda_2 I_2 + ... + \lambda_n I_n\| := \sqrt{|\lambda_1|^2 + |\lambda_2|^2 + ... + |\lambda_n|^2}$.

The algebra $\mathbb{A}_n^m$ contains m maximal ideals

$$\mathcal{I}_u := \left\{ \sum_{r=1,\, r\neq u}^{n} \lambda_r I_r : \lambda_r \in \mathbb{C} \right\}, \quad u = 1, 2, \ldots, m.$$

Consider m linear functionals $f_u : \mathbb{A}_n^m \to \mathbb{C}$ satisfying the equalities

$$f_u(I_u) = 1, \quad f_u(\omega) = 0 \quad \forall\, \omega \in \mathcal{I}_u\,, \quad u = 1, 2, \ldots, m.$$

Fix real k-dimentional subspace $E_k := \{\zeta = x_1e_1 + x_2e_2 + ... + x_ke_k \,:\, x_1, ..., x_k \in \mathbb{R}\} \subset \mathbb{A}_n^m$, where vectors $e_1, ..., e_k$ are linearly independent over the field $\mathbb{R}$. We will impose following restriction on the choice of the linear span E_k:

$$\{f_u(\zeta) : \zeta \in E_k\} = \mathbb{C}\,, \qquad u = 1, 2, \ldots, m\,, \tag{1}$$

i.e., the images of the set E_k under all mappings f_u must be the whole complex plane.

The intersection of some maximal ideal $\mathcal{I}_u$, $u \in [1, m] \cap \mathbb{N}$, of the algebra $\mathbb{A}_n^m$ with the linear space E_k is a certain plane L_u of dimension $k-2$. The preimage in E_k of an arbitrary point $\xi \in \mathbb{C}$ under the mapping f_u is the plane $L_u^\zeta := \{\zeta + \tau \,:\, \tau \in L_u\}$, where ζ is some element of E_k such that $\xi = f_u(\zeta)$. Obviously, the plane L_u^ζ is parallel to the plane L_u.

Definition 1. We say that a function $\Phi : \Omega \to \mathbb{A}_n^m$ satisfies the condition $K'''_{\mathbb{A}_n^m, E_k}$ at the point $\zeta \in \Omega \subset E_k$, if there exists an element $\Phi_*(\zeta) \in \mathbb{A}_n^m$ such that the equality

$$\lim_{\delta \to 0+0} \left(\Phi(\zeta + \delta h) - \Phi(\zeta)\right) \delta^{-1} = h\Phi_*(\zeta) \tag{2}$$

holds for k distinct vectors $h_1, h_2, \ldots, h_k$ that form a basis in the space E_k, and also such that for each mapping f_u, $u \in [1, m] \cap \mathbb{N}$, there are exactly two of them, whose images under the mapping f_u are noncollinear in $\mathbb{C}$.

Theorem 2. *Let a domain $\Omega \subset E_k$ have the property that its intersections with the planes L_u^ζ are connected, where $\zeta \in \Omega$, $u = \overline{1, m}$, and let a function $\Phi : \Omega \to \mathbb{A}_n^m$, continuous in Ω, satisfy the condition $K'''_{\mathbb{A}_n^m, E_k}$ at all points $\zeta \in \Omega$, except for at most a countable set of points. Then the function Φ is monogenic in the domain Ω.*

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Basic aspects of some special geometry of tangent bundle for spaces of affine connection

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We consider a space of affine connection A^n $(n > 2)$ of the class of smooth $C^r(r > 1)$ with the object of affine connection Γ that is determined by components $\Gamma^h_{ij}(x^1, x^2, ..., x^n),\ h, i, j = \overline{1, n}$ according to the every local coordinate system $(x^i), i = \overline{1, n}$ [1].

Some special broadened affine connection $\tilde{\Gamma}$ is built. It has the components $\tilde{\Gamma}^h_{ij}(x^1, x^2, ..., x^n; y^1, y^2, ..., y^n)$, $h, i, j = \overline{1, n}$, according to the every local coordinate system,

$$\tilde{\Gamma}^h_{ij}(x^1, x^2, ..., x^n; y^1, ..., y^n) = \Gamma^h_{ij}(x^1, x^2, ..., x^n) + T^h_{(ij)\alpha} y^\alpha, h, i, j, \alpha = \overline{1, n}.$$

Here

$$T^h_{ij\alpha} = \delta^h_i R_{j\alpha} - R^h_{ij\alpha},$$

δ^h_i are the Kronecker delta symbols, $R_{j\alpha}$ are the Ricci symbols, $R^h_{ij\alpha}$ are the Riemann symbols of the space A^n, parentheses denote symmetrization of the included indices without devision. In the contrast to components of the object Γ components of the object $\tilde{\Gamma}$ depend from $2n$ variables. It is naturally to consider them in the mentioned way to be defined in the space of the tangent bundle $T(A^n)$. At the same time, proceed from the number of components $\tilde{\Gamma}^h_{ij}(x^1, x^2, ..., x^n; y^1, y^2, ..., y^n)$, the broadened connection $\tilde{\Gamma}$ may be considered as a connection in A^n that depends not only on the coordinates of the current point but also from the components of the tangent vector in it. This means that the connection $\tilde{\Gamma}$ is in some sense similar to connections of Cartan and Bervald of Finsler geometry.

On the base of the connection $\tilde{\Gamma}$, by the help of the operation of the type of full lift, the connection $c\tilde{\Gamma}$ is built.

Its components $c\tilde{\Gamma}^h_{ij}(x^1, x^2, ..., x^n; y^1, y^2, ..., y^n)$ are the next.

$$c\tilde{\Gamma}^h_{ij}(x^1, x^2, ..., x^n; y^1, y^2, ..., y^n) = \tilde{\Gamma}^h_{ij}(x^1, x^2, ..., x^n; y^1, y^2, ..., y^n),\ h, i, j = \overline{1, n};$$

$$c\tilde{\Gamma}^h_{ij}(x^1, x^2, ..., x^n; y^1, y^2, ..., y^n) = \frac{\partial \tilde{\Gamma}^{h-n}_{ij}(x^1, x^2, ..., x^n; y^1, y^2, ..., y^n)}{\partial x^\alpha} y^\alpha,\ h = \overline{n+1, 2n}, i, j, \alpha = \overline{1, n};$$

$$c\tilde{\Gamma}^h_{ij}(x^1, x^2, ..., x^n; y^1, y^2, ..., y^n) = \tilde{\Gamma}^{h-n}_{ij-n}(x^1, x^2, ..., x^n; y^1, y^2, ..., y^n),\ h, j = \overline{n+1, 2n},\ i = \overline{1, n};$$

$$c\tilde{\Gamma}^h_{ij}(x^1, x^2, ..., x^n; y^1, y^2, ..., y^n) = \tilde{\Gamma}^{h-n}_{i-nj}(x^1, x^2, ..., x^n; y^1, y^2, ..., y^n),\ h, i = \overline{n+1, 2n},\ j = \overline{1, n};$$

$$c\tilde{\Gamma}^h_{ij}(x^1, x^2, ..., x^n; y^1, y^2, ..., y^n) = 0,\ i = \overline{n+1, 2n}, h, j = \overline{1, n};$$

$$c\tilde{\Gamma}^h_{ij}(x^1, x^2, ..., x^n; y^1, y^2, ..., y^n) = 0,\ i, j = \overline{n+1, 2n}, h = \overline{1, n};$$

$$c\tilde{\Gamma}^h_{ij}(x^1, x^2, ..., x^n; y^1, y^2, ..., y^n) = 0,\ i = \overline{n+1, 2n}, h, i, j = \overline{n+1, 2n}.$$

By the help of the connection $c\tilde{\Gamma}$ the covariant differentiation is introduced in $T(A^n)$, by the traditional way the concept of geodesic line is defined. Interdependences between the concepts of a geodesic line in the space A^n according to the connection Γ, the concept of a geodesic line in this space A^n according to the connection $\tilde{\Gamma}$ and the concept of a geodesic line in the space $T(A^n)$ according to the connection $\tilde{\Gamma}$ are investigated.

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НТБ ОНТУ

Homotopy structure of orbits of smooth functions lifted to an orientable double cover

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Let M be a surface (smooth compact 2-manifold) and $f\colon M \to \mathbb{R}$ be a smooth function. By smooth we mean $\mathcal{C}^\infty$. We study the homotopy types of orbits $\mathcal{O}(f)$ with respect to the natural right action of the smooth diffeomorphisms group on the space of smooth functions $\mathcal{C}^\infty(M,\mathbb{R})\times\mathcal{D}(M) \to \mathcal{C}^\infty(M,\mathbb{R})$, given by $(f,h)\mapsto f\circ h$.

Consider the subclass $\mathcal{F}(M,\mathbb{R}) \subset \mathcal{C}^\infty(M,\mathbb{R})$ of functions that, in particular, contains Morse functions, which are dense in $\mathcal{C}^\infty(M,\mathbb{R})$. It is well-known (see [2]) that, for each $f \in \mathcal{F}(M,\mathbb{R})$, the homotopy types of $\mathcal{O}(f)$ are weak, i.e., they are completely described by the collection of the homotopy groups $\pi_k\mathcal{O}(f)$, $k \geq 1$. Only a few cases yield non-trivial higher homotopy groups $\pi_k\mathcal{O}(f)$, $k \geq 2$, while the structure of the fundamental group $\pi_1\mathcal{O}(f)$ is highly non-trivial and is of the main interest.

Let $\tilde{N}$ be an orientable surface with a smooth orientation-reversing involution $\tau\colon \tilde{N} \to \tilde{N}$ without fixed points, and $N = \tilde{N}/\tau$ be the quotient, a non-orientable surface. Then there is a smooth double covering map $p\colon \tilde{N} \to N$, and the lifted function $\tilde{f} = f\circ p$, which lies in the class $\mathcal{F}(\tilde{N},\mathbb{R})$. One can uniquely map each $h \in \mathcal{D}(N)$ to the orientation-preserving lift $\tilde{h} \in \mathcal{D}(\tilde{N})$ such that $p\circ\tilde{h} = h\circ p$. This lifting induces a well-defined mapping of orbits $\theta\colon \mathcal{O}(f) \to \mathcal{O}(\tilde{f})$.

Theorem 1. *Map θ is π_1-injective, i.e., induces a group monomorphism $\pi_1\mathcal{O}(f) \to \pi_1\mathcal{O}(\tilde{f})$.*

Theorem 2. *Assume Euler characteristic $\chi(N) < 0$, and function $f \in \mathcal{F}(N,\mathbb{R})$.*

(1) There exists (see e.g. [1]) a finite collection $\{S_\alpha\}$ of subsurfaces of N whose elements are diffeomorphic either to disks D_i, cylinders C_j, or Mobius bands M_k, such that

$$\pi_1\mathcal{O}(f) \cong \prod_\alpha \pi_1\mathcal{O}(f|_{S_\alpha}) \cong \prod_i \pi_1\mathcal{O}(f|_{D_i}) \times \prod_j \pi_1\mathcal{O}(f|_{C_j}) \times \prod_k \pi_1\mathcal{O}(f|_{M_k}).$$

(2) There exists a finite collection $\{\tilde{S}_\beta\}$ of subsurfaces of $\tilde{N}$ whose elements are diffeomorphic either to disks $\tilde{D}_u$, or cylinders $\tilde{C}_v$. Moreover, there is a subcollection of cylinders $\{\tilde{C}_k\} \subset$

$\{\tilde{C}_u\}$ which are in one-to-one correspondence with the collection of Mobius bands $\{M_k\}$, such that

$$\pi_1\mathcal{O}(\tilde{f}) \cong \prod_\beta \pi_1\mathcal{O}(f|_{\tilde{S}_\beta}) \cong \prod_u \pi_1\mathcal{O}(\tilde{f}|_{\tilde{D}_u}) \times \prod_v \pi_1\mathcal{O}(\tilde{f}|_{\tilde{C}_v})$$
$$\cong \prod_i (\pi_1\mathcal{O}(f|_{D_i}))^2 \times \prod_j (\pi_1\mathcal{O}(f|_{C_j}))^2 \times \prod_k \pi_1\mathcal{O}(\tilde{f}|_{\tilde{C}_k}).$$

In particular, $\#\{\tilde{D}_u\} = 2\#\{D_i\}$ and $\#\{\tilde{C}_v\} = 2\#\{C_j\} + \#\{M_k\}$.

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On average elements of weighted sets

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In mathematics, means of numerical collections represent the values that are the most typical in some sense. There are three classical means, namely the arithmetic mean (AM), the geometric mean (GM) and the harmonic mean (HM), and also the Lehmer and Hölder means, etc.

Now we proceed to the main definitions that introduce a new notion of average elements for finite weighted sets. For clarity, we consider the case of the arithmetic average, noting that other ones are treated similarly.

A *weighted set* is a pair (X, w), where X is a set and w is a mapping from X into the set $\mathbb{R}^+$ of nonnegative real numbers; $w(x)$ is called the *weight* of x.

Let (X, w) be a finite weighted set with $X = \{x_1, \dots, x_n\}$, and let x_w be the arithmetic mean of the numbers $w(x_1), \dots, w(x_n)$. Fix an element $x_i \in X$ such that the modulus of the difference $w(x_i) - x_w$ is minimal. In this case, we say that x_i is an *arithmetic average element* of the weighted set (X, w). Note that there may be several such elements.

We identify the singletons with the elements themselves. If the indicated difference is positive (resp. negative), the set of all average elements is called *right* (resp. *left*). Finally, if this difference is equal to 0, the set is called *exact*.

In this paper, we consider weighted sets of a special form when the sets are posets and the weights relate to the transitivity coefficients.

Let S be a finite poset and $S^2_\prec := \{(x, y) \mid x, y \in S, x \prec y\}$. Elements x and y are called *neighboring* if $(x, y) \in S^2_\prec$ and there is no z satisfying $x \prec z \prec y$. Let us denote by $n_w = n_w(S)$ the cardinality of the set $S^2_\prec$, and by $n_e = n_e(S)$ the number of pairs (x, y) of neighboring elements of S. The ratio $k_t = k_t(S)$ of the numbers $n_w - n_e$ and n_w is called the *coefficient of transitivity* of S ($k_t := 0$ for $n_w = 0$) [1].

In the language of the Hasse diagram $H(P)$ of S, n_e is equal to the number of all its edges, and n_w to the number of all paths directed bottom-up, up to parallelism (i.e., up to paths with the same start and the same terminal vertices).

We call a pair (S, k_t) a *weighted poset respectively the transitivity coefficient.* And an arbitrary class $\mathcal{C}$ of finite posets can be considered as the weighted class $(\mathcal{C}, k_t)$.

As an example, we consider the class of positive posets.

A finite poset $S \not\ni 0$ is called *positive* if so is its *Tits quadratic form* $q_S(z) : \mathbb{Z}^{\{0\}\cup S} \to \mathbb{Z}$, given by the following equality (see [2]):

$$q_S(z) = z_0^2 + \sum_{i\in S} z_i^2 + \sum_{i\prec j,\, i,j\in S} z_i z_j - z_0 \sum_{i\in S} z_i.$$

A positive poset S is called *serial* if there is an infinite increasing sequence $S \subset S^{(1)} \subset S^{(2)} \subset \dots$ with positive terms, and *nonserial* otherwise.

Serial and nonserial positive posets were first classified in [3, 4] (see also the more accessible paper [5]). The result was obtained by the minimax equivalence method [6] (for more detail, see [7]).

Now we proceed to the main theorems.

A poset T is called *dual* to a poset S and is denoted by S^{op} if it has the same elements and the inverse partial order relation. T is called *anti-isomorphic* to S if it is isomorphic to S^{op}.

We study only classes of posets closed under both isomorphism and duality (anti-isomorphism). If X is such a class, we consider its elements up to isomorphism and up to both isomorphism and duality. In the first case, the set of the corresponding classes is denoted by $[X]$, and in the second case by $[X]_{op}$. A complete system of representatives of $[X]$ (resp. $[X]_{op}$) is denoted by $[X]^0$ (resp. $[X]^0_{op}$).

Theorem 1. *Let $\mathcal{P}_0$ denote the class of the minimal nonserial positive posets. Then the following holds:*

(*a*) *there exists exactly one arithmetic average element of $\left([\mathcal{P}_0]^0_{op}, k_t\right)$, namely the one which is nonconnected and non-self-dual;*

(*b*) *there exist exactly two arithmetic average elements of $([\mathcal{P}_0]^0, k_t)$, namely the two mutually dual elements which are nonconnected and non-self-dual;*

(*c*) *the set of average elements is left in case* (*a*) *and right in case* (*b*).

Theorem 2. *Let $\mathcal{P}_0$ be as in Theorem 1. Then the sets of geometric average elements of $\left([\mathcal{P}_0]^0_{op}, k_t\right)$ and $([\mathcal{P}_0]^0, k_t)$ consist of all elements whose height is less than two. They contain, respectively, three and four elements, and both are exact.*

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On Generalized Graham–Kohr Extension Operators, Loewner Chains, and Janowski Subordination in the Unit Ball

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We study generalized Graham–Kohr type extension operators associated with normalized locally univalent functions on the unit disk $U \subset \mathbb{C}$ and their extensions to holomorphic mappings on the unit ball $B^n \subset \mathbb{C}^n$. These operators play an important role in extending geometric properties from one complex variable to several complex variables and include, as a particular case, the classical Roper–Suffridge operator.

Let f be a locally univalent function defined on the unit disk $\mathbb{U}$, and let $\gamma_n \in [0,1]$ and $\omega_n \in \left[0, \frac{1}{2}\right]$. We consider the family of extension operators

$$\Theta_{n,\gamma_n,\omega_n}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^{\gamma_n} \left(f'(z_1)\right)^{\omega_n} z' \right),$$

where $z = (z_1, z') \in \mathbb{C}^n$, $z' = (z_2, \ldots, z_n)$, and $n \geq 2$. When $\omega_n = \frac{1}{2}$ and $\gamma_n = 0$, the above operator reduces to the classical Roper–Suffridge extension operator. We show that $\Theta_{n,\gamma_n,\omega_n}$ maps the class of spirallike functions of type β (denoted by $\hat{S}_\beta$) into the class $S^0(\mathbb{B}^n)$ of mappings having parametric representation on $\mathbb{B}^n$. Furthermore, we prove that if f is a normalized univalent Bloch function on $\mathbb{U}$, then $\Theta_{n,\gamma_n,\omega_n}(f)$ is a Bloch mapping on the unit ball $\mathbb{B}^n$.

We further investigate a more general extension operator introduced by M. Aron. For parameters $\alpha \in [0,1]$, $\beta \in [0, \frac{1}{2}]$, and $\gamma \geq 1$, we consider

$$\Psi^{\gamma}_{n,\alpha,\beta}(f)(z) = \left(f(z_1), \tilde{z} \left[\frac{f(z_1)}{z_1}\right]^{\alpha} \left[1 + \frac{1}{\gamma}\left(\frac{z_1 f'(z_1)}{f(z_1)} - 1\right)\right]^{\beta} \right),$$

where $z = (z_1, \tilde{z}) \in \mathbb{C}^n$, $\tilde{z} = (z_2, \ldots, z_n)$, and $n \geq 2$. We investigate preservation properties of this operator for several geometric subclasses of univalent functions. In particular, for real numbers a, b satisfying $|1-a| < b \leq a$, we establish sufficient conditions on the parameters α, β, γ ensuring that

$$\Psi^{\gamma}_{n,\alpha,\beta}(S^*(a,b)) \subset S^*(a,b,B^n), \qquad \Psi^{\gamma}_{n,\alpha,\beta}(AS^*(a,b)) \subset AS^*(a,b,B^n).$$

In addition, for suitable choices of the parameters, we establish subordination results for subclasses of starlike functions.

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Exact solutions of nonlinear dispersive evolution equations

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Many model equations for various wave processes can be reduced to nonlinear dispersive evolution equations such as the classical Korteweg–de Vries equation, the modified Korteweg–de Vries equation, and the Kawahara equation, as well as their generalizations, which often involve variable coefficients. This explains the great interest of researchers in finding new and applying already known methods for constructing exact solutions to such generalized equations. At the same time, most of the proposed methods lead to equivalent forms of already known solutions, primarily because the equivalence of models and corresponding solutions is not systematically investigated. This widespread oversight in finding solutions to differential equations was thoroughly analyzed in [1]. Conversely, the application of equivalence transformations serves as an effective tool for finding exact solutions, solving group classification problems, and investigating the integrability of nonlinear equations of mathematical physics with variable coefficients [1, 2, 3]. So, the equivalence method can be proposed as a method of choice for a wide range of such problems.

In this talk, we discuss the use of equivalence transformations for constructing exact solutions of nonlinear dispersive evolution equations of KdV type with variable coefficients. The respective results used as illustrative examples have been published, in particular, in [2, 4, 5, 6, 7].

Acknowledgements. This work was supported by a grant from the Simons Foundation (SFI-PD-Ukraine-00014586, O.O.V. and A.Yu.Zh.).

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Simplicity of Lie algebras of vector fields

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Let K be an algebraically closed field of characteristic zero, let $A = K[x_1, \dots, x_n]$ be the polynomial ring in n variables, and let $R = K(x_1, \dots, x_n)$ be the field of rational functions. A K-derivation on the ring A is a K-linear map $D : A \to A$ such that $D(fg) = D(f)g + fD(g)$ for all $f, g \in A$. Every K-derivation can be uniquely written in the form:

$$D = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}, \quad f_i = f_(x_1, \dots, x_n) \in A,$$

where $\frac{\partial}{\partial x_i}$ are the usual partial derivatives. The K-vector space of all derivations of the ring A is a Lie algebra over K with respect to the Lie bracket $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$. We denote this Lie algebra by $W_n(K)$. It follows from the above that $W_n(K)$ is a free A-module with the standard basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. The Lie algebra $W_n(K)$ is of great interest since, from the geometric point of view, $W_n(K)$ consists of all polynomial vector fields on K^n. The structure of subalgebras of the Lie algebra $W_n(K), n \geq 2$ has been studied by many authors, but many open problems remain in this direction; see, for example, [5], [1]. The subalgebras of the Lie algebra $W_1(K)$ are described in [2]. In [4], it was shown that there are close connections between ideals of the the polynomial ring A and subalgebras of the Lie algebra $W_n(K)$. Especially interesting are subalgebras of the Lie algebra $W_n(K)$ which are simultaneously submodules of the free A-module $W_n(K)$. We call such subalgebras polynomial subalgebras (see also [3]). One of the main results of this paper states that every Lie algebra $L \subseteq W_n(K)$ having no nonzero proper L-invariant ideals is a simple Lie algebra. We also point out an another source of simple subalgebras, namely maximal subalgebras of rank $k < n$ of the Lie algebra $W_n(K)$. Here the rank $\mathrm{r}kL$ of a subalgebra $L \subseteq W_n(K)$ is, as usual, dimension $\dim_R RL$). Note that maximal subalgebras of rank $< n$ are automatically polynomial subalgebras. Using results of the paper [4] we obtain the following result.

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Lemma 1. *Let $X = V(I)$, $X \subset K^n$ be an affine algebraic variety with the defining ideal I, and let $p \in K^n \setminus X$ be a point. Then, for an arbitrary vector $\vec{u} = (u_1, \ldots, u_n) \in K^n$, there exists a derivation $D = \sum_{i=1}^{n} f_i(x_1, \ldots, x_n)\frac{\partial}{\partial x_i} \in W_n$ such that:*

(1) $D(I) \subseteq I$;

(2) $v_D(p) = (u_1, \ldots, u_n)$, where v_D is the vector field on K^n corresponding to D.

Theorem 2. *Let M be a maximal subalgebra of the Lie algebra W_n with $rkM = k < n$. Then*

(1) *the only M-invariant ideals of the polynomial ring A are A and* 0.

(2) *M is a simple subalgebra of the Lie algebra W_n.*

The situation for maximal polynomial subalgebras of rank n is opposite as the next statement shows.

Theorem 3. *Let M be a polynomial maximal subalgebra of* rank n *from W_n. Then*

(1) $M = \{D \in W_n \mid D(I) \subseteq I\}$ for an ideal I of the ring A.

(2) The subalgebra I^2W_n is a nonzero proper ideal of M, so M is a non-simple Lie algebra.

We use here properties of modules of maximal rank over the polynomial ring A. If M is such a module then $I = \{f \in A \mid fW_n(K) \subseteq M\}$ is a proper nonzero ideal of the ring A.

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Geometry of encoding of numbers by means of a redundant alphabet in problems of function theory with fractal properties

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We consider representations of numbers from the interval $[0; \frac{r}{s-1}]$ with a natural base $s \geq 2$ and alphabet $A_r \equiv \{0, 1, ..., r\}$, where $s \leq r \leq 2s - 2$, namely $x = \Delta^{r_s}_{\alpha_1\alpha_2...\alpha_n...} \equiv \frac{\alpha_1}{s} + \frac{\alpha_2}{s^2} + ... + \frac{\alpha_n}{s^n} + ...$

The case $r = 2s - 2$ requires special attention, since for $r > 2s - 2$ all numbers possess a continuum of distinct representations, whereas for $s \leq r < 2s - 2$ there exists a continuum set of numbers having unique representations. In the boundary case $r = 2s - 2$ there is a unique subcase $s = 2 = r$, in which the points of a countable set have countably many representations, while all remaining points have continuum many representations.

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Lemma 1. *If $s = 2 = r$, then the cylinders $\Delta^{r_s}_{c_1 \dots c_{m-1} i} = [c; \frac{1}{s^m} + c]$, $\Delta^{r_s}_{c_1 c_2 \dots c_{m-1}[i+1]} = [c + \frac{1}{s^m}; \frac{2}{s^m} + c]$, $c = \sum\limits_{i=1}^{m-1} \frac{c_i}{2^i}$, have the specific intersection property $\Delta^{r_s}_{c_1 \dots c_{m-1} i} \cap \Delta^{r_s}_{c_1 \dots c_{m-1}[i+1]} = \Delta^{r_s}_{c_1 \dots c_m i 2} = \Delta^{r_s}_{c_1 \dots c_{m-1}[i+1]0}$.*

The specific overlap structure of neighboring cylinders described in Lemma 1 is another distinctive feature of this case.

Henceforth, let $s = 2 = r$.

Theorem 2. *The function f, defined by the equation $f(x = \Delta^3_{\alpha_1 \dots \alpha_n \dots}) = \Delta^{r_s}_{\alpha_1 \dots \alpha_n \dots}$, is continuous at points having a unique representation in the classical ternary numeral system and discontinuous at ternary-rational points: $\Delta^3_{c_1 \dots c_{m-1} c_m (0)} = \Delta^3_{c_1 \dots c_{m-1}[c_m - 1](2)}$. Moreover, it possesses fractal level sets.*

Theorem 3. *The range of the complex-valued function $\varphi(t) = \sum\limits_{n=1}^{\infty} 2^{-n} \varepsilon_{\alpha_n(t)}$ of a real argument $t = \Delta^3_{\alpha_1 \alpha_2 \dots \alpha_n \dots}$, where ε_0, ε_1, ε_2 are the cubic roots of unity, is a spiderweb-type curve having self-similar fractal dimension $\log_2 3$, which coincides with its Hausdorff–Besicovitch dimension.*

Theorem 4. *Let (ξ_n) be a sequence of independent random variables taking values 0, 1, and 2 with probabilities p_0, p_1, and p_2 respectively, where $\max\{p_0, p_1, p_2\} \neq 1$. Then the random variable $\xi = \Delta^{r_s}_{\xi_1 \xi_2 \dots \xi_n \dots}$ has: 1) an absolutely continuous distribution if and only if $p_1 = \frac{1}{2}$ or $p_0 = p_2 = \frac{1}{2}$; 2) a purely singular distribution in all remaining cases.*

We note that an explicit expression for the distribution function of the random variable ξ is still unknown. In this case, the problem is of a combinatorial nature.

Theorem 5. *Let (τ_n) be a sequence of random variables taking values 0, 1, and 2, forming a Markov chain with nonzero initial probabilities p_0, p_1, p_2, and transition probability matrix $||p_{ik}||$. Then the random variable $\tau = \Delta^{r_s}_{\tau_1 \tau_2 \dots \tau_n \dots}$ may have purely discrete, purely singular, or purely absolutely continuous distributions, as well as distributions that are mixtures of discrete and continuous components.*

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On convex bodies with planar projections having rotational symmetries

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Let $n \geq 3$ and let $K \subset \mathbb{R}^n$ be a convex body. For a two-dimensional linear subspace $P \subset \mathbb{R}^n$, let $K|P$ be the orthogonal projection of K onto P. We prove that if for every two-dimensional subspace P, the planar convex body $K|P$ has q-fold rotational symmetry up to translation, then for $q \geq 4$, this forces K to be a ball. The case $q = 3$ is exceptional: non-spherical examples exist.

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Sturm-Liouville problems with a boundary condition depending linearly on an eigenparameter

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Consider the following spectral problem

$$-y'' + q(x)y = \lambda y,\ 0 < x < 1, \tag{1.1}$$

$$y(0)\cos\beta = y'(0)\sin\beta,\ 0 \le \beta < \pi, \tag{1.2}$$

$$(a\lambda + b)y(1) = (c\lambda + d)y'(1), \tag{1.3}$$

where λ is a spectral parameter, $q(x)$ is a real-valued and continuous function over the interval $(0,1)$ and a, b, c, d are real constants with $ad - bc \neq 0$. It is possible to find the eigenfunctions and associated functions corresponding to the eigenvalues λ_n.

The sufficiently large eigenvalues of the boundary value problem (1.1)–(1.3) are real, simple, and form an infinite increasing sequence

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots .$$

Let $\varphi(x,\lambda)$ be a non-zero solution of (1.1)–(1.2). We write the characteristic function, which is analytic, as

$$\omega(\lambda) = (a\lambda + b)y(1,\lambda) - (c\lambda + d)y'(1,\lambda),$$

It follows from the boundary condition (1.3) that a necessary and sufficient condition for λ to be an eigenvalue is that the equation

$$\omega(\lambda) = 0$$

is satisfied. If, in addition, the inequality

$$\omega'(\lambda) \neq 0$$

is satisfied, then we say that λ is a simple eigenvalue.

It is known that the boundary value problem (1.1)–(1.3) has infinitely many eigenvalues, and the following cases are possible:

(a) all eigenvalues are real and simple;
(b) all eigenvalues are real and all, except one double λ_k, are simple;
(c) all eigenvalues are real and all, except one triple λ_k, are simple;
(d) all eigenvalues are simple and all, except a conjugate pair of non-real ones $\lambda_s = \overline{\lambda_r}$, are real
.

The eigenfunctions and associated functions are defined as follows. If λ_k is a double eigenvalue ($\lambda_k = \lambda_{k+1}$) or if λ_k is a triple eigenvalue ($\lambda_k = \lambda_{k+1} = \lambda_{k+2}$), then for the associated function y_{k+1} corresponding to the eigenfunction y_k, the following relations hold:

$$-y''_{k+1} + q(x)y_{k+1} = \lambda_k y_{k+1} + y_k, \tag{1.4}$$

$$y_{k+1}(0)\cos\beta = y'_{k+1}(0)\sin\beta, \tag{1.5}$$

$$(a\lambda_k + b)\, y_{k+1}(1) + a y_k(1) = (c\lambda_k + d)\, y'_{k+1}(1) + c y'_k(1). \tag{1.6}$$

If λ_k is an eigenvalue of multiplicity three, i.e., $\lambda_k = \lambda_{k+1} = \lambda_{k+2}$, then in addition to the first-order associated function y_{k+1} defined by (1.4)-(1.6), there exists a second-order associated function y_{k+2}:

$$-y''_{k+2} + q(x)y_{k+2} = \lambda_k y_{k+2} + y_{k+1}, \tag{1.7}$$

$$y_{k+2}(0)\cos\beta = y'_{k+2}(0)\sin\beta, \tag{1.8}$$

$$(a\lambda_k + b)\, y_{k+2}(1) + a y_{k+1}(1) = (c\lambda_k + d)\, y'_{k+2}(1) + c y'_{k+1}(1). \tag{1.9}$$

Let y^*_{k+1} ,$y^{\#}_{k+1}$ and $y^{\#}_{k+2}$ be functions satisfying the following relations:

$$\left(y^*_{k+1}, y_{k+1}\right) = -(ad-bc)\mathfrak{A}(y^*_{k+1})\mathfrak{A}(y_{k+1})$$

$$\left(y^{\#}_{k+1}, y_{k+2}\right) = -(ad-bc)\mathfrak{A}(y^{\#}_{k+1})\mathfrak{A}(y_{k+2})$$

$$\left(y^{\#}_{k+2}, y_{k+2}\right) = -(ad-bc)\mathfrak{A}(y^{\#}_{k+2})\mathfrak{A}(y_{k+2}).$$

Let us define :

$$\mathfrak{A}(y^*_{k+1}) = \begin{cases} \dfrac{y^*_{k+1}(1)}{c\lambda_k + d} - \dfrac{c y_k(1)}{(c\lambda_k + d)^2}, & \lambda_k \neq -\dfrac{d}{c}, \\ \dfrac{(y^*_{k+1})'(1)}{a\lambda_k + b} - \dfrac{a y'_k(1)}{(a\lambda_k + b)^2}, & \lambda_k = -\dfrac{d}{c}, \end{cases}$$

$$\mathfrak{A}(y^{\#}_{k+1}) = \begin{cases} \dfrac{y^{\#}_{k+1}(1)}{c\lambda_k + d} - \dfrac{c y_k(1)}{(c\lambda_k + d)^2}, & \lambda_k \neq -\dfrac{d}{c}, \\ \dfrac{(y^{\#}_{k+1})'(1)}{a\lambda_k + b} - \dfrac{a y'_k(1)}{(a\lambda_k + b)^2}, & \lambda_k = -\dfrac{d}{c}, \end{cases}$$

$$\mathfrak{A}(y^{\#}_{k+2}) = \begin{cases} \dfrac{y^{\#}_{k+2}(1)}{c\lambda_k + d} - \dfrac{c y^{\#}_{k+1}(1)}{(c\lambda_k + d)^2} + \dfrac{c^2 y_k(1)}{(c\lambda_k + d)^3}, & \lambda_k \neq -\dfrac{d}{c}, \\ \dfrac{(y^{\#}_{k+2})'(1)}{a\lambda_k + b} - \dfrac{a (y^{\#}_{k+1})'(1)}{(a\lambda_k + b)^2} + \dfrac{a^2 y'_k(1)}{(a\lambda_k + b)^3}, & \lambda_k = -\dfrac{d}{c}. \end{cases}$$

Theorem 1. *Assume that all eigenvalues of problem (1.1)–(1.3) are real and simple. Then the system*

$$\{y_n\} \quad (n = 0, 1, \ldots;\ n \neq l), \tag{1.10}$$

where l is a nonnegative integer, is a basis of the space $L_p(0,1)$ $(1 < p < +\infty)$.

Theorem 2. *If λ_k is an eigenvalue of multiplicity two, then the system*

$$\{y_n\} \quad (n = 0, 1, \ldots;\ n \neq k+1), \tag{1.11}$$

is a basis of the space $L_p(0,1)$ $(1 < p < +\infty)$.

Theorem 3. *If λ_k is an eigenvalue of multiplicity two, then the system*

$$\{y_n\} \ \ (n = 0,\ 1, \ldots; n \neq k), \tag{1.12}$$

is a basis of the space $L_p(0,1)$ $(1 < p < +\infty)$ if and only if $\mathfrak{A}(y^*_{k+1}) \neq 0$.

Theorem 4. *If λ_k is an eigenvalue of multiplicity two, then the system*

$$\{y_n\} \ \ (n = 0,\ 1, \ldots; n \neq l), \tag{1.13}$$

where $l \neq k,\ k+1$ is a non-negative integer,is a basis of the space $L_p(0,1)$ $(1 < p < +\infty)$.

Theorem 5. *If λ_k is an eigenvalue of multiplicity three, then the system*

$$\{y_n\} \ \ (n = 0,\ 1, \ldots; n \neq k+2), \tag{1.14}$$

is a basis of the space $L_p(0,1)$ $(1 < p < +\infty)$

Theorem 6. *If λ_k is an eigenvalue of multiplicity three, then the system*

$$\{y_n\} \ \ (n = 0,\ 1, \ldots; n \neq k+1), \tag{1.15}$$

is a basis of the space $L_p(0,1)$ $(1 < p < +\infty)$ if and only if $\mathfrak{A}(y^{\#}_{k+1}) \neq 0$.

Theorem 7. *If λ_k is an eigenvalue of multiplicity three, then the system*

$$\{y_n\} \ \ (n = 0,\ 1, \ldots; n \neq k), \tag{1.16}$$

is a basis of the space $L_p(0,1)$ $(1 < p < +\infty)$ if and only if $\mathfrak{A}(y^{\#}_{k+2}) \neq 0$.

Theorem 8. *If λ_k is an eigenvalue of multiplicity three, then the system*

$$\{y_n\} \ \ (n = 0,\ 1, \ldots; n \neq l), \tag{1.17}$$

where $l \neq k,\ k+1,\ k+2$ is a non-negative integer, is a basis of the space $L_p(0,1)$ $(1 < p < +\infty)$.

Theorem 9. *Let λ_r and $\lambda_s = \overline{\lambda_r}$ be a pair of complex conjugate non-real eigenvalues. Then each of the systems*

$$\{y_n\} \quad (n = 0, 1, \ldots;\ n \neq r), \tag{1.18}$$

and

$$\{y_n\} \quad (n = 0, 1, \ldots;\ n \neq l), \tag{1.19}$$

where l is a non-negative integer with $l \neq r, s$, is a basis of the space $L_p(0,1)$ $(1 < p < +\infty)$.

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Linearization of Hamilton–Jacobi Equations via Curved Manifolds of Schwartz Waves and Schrödinger Dynamics

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This contribution proposes a geometric and Schwartz-linear reinterpretation of the relationship between Hamilton–Jacobi theory and Schrödinger quantum dynamics. The central idea is that suitable Hamilton principal functions generate, through their quantum exponentials, curved manifolds of Schwartz waves whose Schwartz-superpositional completion produces exact quantum dynamics. Starting from admissible Hamilton–Jacobi families $S = (S_\lambda)_\lambda$, one considers the associated principal exponential family $v_\lambda = \exp(iS_\lambda/\hbar)$. Under suitable spectral admissibility assumptions, these exponentials form continuous or discrete Schwartz spectral bases for quantum Hamiltonians. The Schrödinger equation then appears as the exact linear Schwartz-superpositional extension of Hamilton–Jacobi spectral geometry. The framework simultaneously covers: free particles; relativistic de Broglie waves; particles on compact manifolds; standing-wave systems; harmonic oscillators; continuous spectra; discrete spectra; and finite spectral blocks. The theory also leads to a generalized Schwartz–von Neumann extension principle, according to which classical observables acting on certainty spectral germs extend uniquely to continuous Schwartz-linear operators on the corresponding quantum distribution spaces. The resulting picture suggests that quantum dynamics may be interpreted as the Schwartz-superpositional completion of an underlying geometry of classical certainty germs generated by admissible Hamilton–Jacobi structures. One of the oldest conceptual problems in mathematical physics concerns the relationship between classical Hamilton–Jacobi theory and quantum wave dynamics. Historically, the Hamilton–Jacobi equation provided a geometric and variational description of classical mechanics through action functions and characteristic trajectories, while Schrödinger quantum mechanics introduced linear wave evolution in Hilbert spaces. The de Broglie relation $\psi = \exp(iS/\hbar)$ already suggested a profound connection between these two theories. However, in standard formulations, the relationship often remains local, semiclassical, asymptotic, or purely heuristic. The present work develops a different perspective based on Schwartz linear algebra, tempered distributions, and spectral superposition theory. The central observation is that Hamilton principal functions may generate exact Schwartz spectral families whose superpositions solve the Schrödinger equation globally and rigorously. The geometric mechanism may be summarized as follows:

HJ principal functions $\to$ principal exponentials $\to$ Schwartz spectral families $\to$ quantum dynamics.

In the canonical relativistic case, one starts from Minkowski covectors $p \in M_4^*$ and their associated affine Hamilton principal functions $S_p(x) = \langle p, x\rangle$. Their exponentials are precisely the relativistic de Broglie waves $\eta_p(x) = \exp\left(\frac{i}{\hbar}\langle p, x\rangle\right)$. The flat affine geometry of Minkowski bras thus generates a curved manifold of principal waves inside tempered-distribution state spaces. The Schrödinger equation appears after imposing the relativistic Hamilton–Jacobi compatibility relation $-\partial_t S_p = E(p)$. The resulting spectral geometry is then extended by Schwartz superposition. This viewpoint naturally generalizes: from continuous to discrete spectra; from free particles to bounded systems; from single waves to finite spectral blocks; and from affine action geometries to curved and logarithmic action structures. The present contribution focuses especially on the geometric meaning of these constructions and on the emerging concept of certainty spectral germs.

The theory is developed inside the framework of Schwartz linear algebra and tempered distribution spaces. Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz test-function space and $\mathcal{S}'(\mathbb{R}^n)$ its tempered dual. A Schwartz family $v = (v_\lambda)_{\lambda\in\Lambda}$ is a family of tempered distributions such that the coefficient transform $\phi \mapsto \langle v, \phi\rangle$ has suitable Schwartz regularity properties. If the family forms a Schwartz basis, every state admits a superpositional representation $u = \int_\Lambda a\, v$, where a is a suitable coefficient distribution. This provides the appropriate framework for generalized spectral expansions. The theory introduces admissible Hamilton–Jacobi families: $S = (S_\lambda)_\lambda$. Their principal exponentials are $v_\lambda = \exp(iS_\lambda/\hbar)$. A family S is H-spectrally admissible if: the exponential v forms a Schwartz basis; the generated space is invariant under the quantum Hamiltonian $\hat{H}$; each exponential is an eigenvector: $\widehat{H}v_\lambda = E(\lambda)v_\lambda$. Under these assumptions, every admissible superposition $\psi = \int a\, v$ satisfies the Schrödinger equation $i\hbar\partial_t\psi = \widehat{H}\psi$. The framework also yields a generalized Schwartz–von Neumann extension principle. Given a classical observable $f : \Lambda \to \mathbb{R}$, its spectral extension is the unique continuous Schwartz-linear operator satisfying $\widehat{f}(v_\lambda) = f(\lambda)v_\lambda$. This extends the ordinary diagonal spectral construction to generalized continuous and distributional spectral geometries. A central notion emerging from the theory is the concept of certainty spectral germ. The certainty germ is the elementary spectral object generating the quantum dynamics. Its geometry depends on the physical system under consideration. Examples include: four-momenta for relativistic free particles; quantized angular momenta for particles on a circle; paired opposite momenta for standing-wave systems; quantized action shells for harmonic oscillators. Thus, quantum state spaces appear as Schwartz-superpositional completions of classical certainty-germ geometries. The theory shows that Schrödinger quantum dynamics may be interpreted as the linearization of Hamilton–Jacobi spectral geometry through principal exponentials. More precisely, nonlinear Hamilton–Jacobi structures generate curved manifolds of Schwartz waves inside distribution spaces, while the Schrödinger equation governs their linear Schwartz-superpositional evolution. Several different spectral geometries are shown to fit into the same mechanism. **Continuous spectral geometry.** For free relativistic particles, the certainty germs are Minkowski covectors $p \in M_4^*$. The associated Hamilton principal functions are $S_p(x) = \langle p, x\rangle$, and the principal waves are $\eta_p(x) = \exp(iS_p(x)/\hbar)$. The relativistic Hamilton–Jacobi relation reduces the admissible spectral geometry to the mass shell $\langle p, p\rangle = -m_0^2c^2$. The Schrödinger equation then becomes the exact superpositional extension of the corresponding principal-wave geometry. **Compact spectral geometry.** For particles on a ring, the certainty germs are quantized angular momenta $p_n = \hbar n$. The associated actions are $S_n(t, \theta) = \hbar n\theta - E_n t$. Their exponentials $\exp(iS_n/\hbar)$ form the Fourier basis on the circle. The discreteness arises from topological single-valuedness of the principal exponentials. **Boundary spectral geometry.** For the particle in a box, the elementary certainty structure is no longer a single momentum but a paired spectral block: $(p, -p)$. The physical standing waves arise as admissible antisymmetric combinations of the two opposite traveling principal exponentials. This shows that admissible Hamilton–Jacobi structures may possess internal finite-dimensional spectral geometry before the final quantum basis is selected. The theory suggests the following geometric classification.

System	Certainty Germ	Spectral Type	Action Geometry
Free particle	four-momentum	continuous	real affine
Particle on a ring	angular momentum	discrete	real periodic
Particle in a box	paired momenta	discrete block	paired affine

These examples show that the same Schwartz-superpositional mechanism survives under: translational geometry; compact topology; boundary constraints; standing-wave selection; localization;

and nodal spectral geometry. The framework therefore extends far beyond ordinary Fourier analysis. The free particle corresponds only to the flat translational realization of a much more general Hamilton–Jacobi spectral mechanism.

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A criterion for finitely many intervals in achievement sets of random power series with restricted alphabet

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Let ξ_i be independent identically distributed random variables with values in the alphabet $A = \{a_0 = 0, a_1, \dots, a_p = r \mid a_i \in \mathbb{N} \cup \{0\}, a_i > a_{i-1}\}$. We study the set of values of the random variable

$$\xi = \sum_{k=1}^{\infty} \xi_k s^{-k},$$

where $s \geq 2$ is a fixed natural number. We denote this set by $\mathcal{E}(\xi)$.

Definition 1. By a *cylindrical interval* of rank n of the half-open interval $[0, 1)$ we mean the half-open interval

$$\widetilde{\Delta}_{\alpha_1\alpha_2\dots\alpha_n} := \left[\sum_{k=1}^{n} \frac{\alpha_k}{s^k}, \sum_{k=1}^{n} \frac{\alpha_k}{s^k} + \frac{1}{s^n}\right),$$

where $\alpha_k \in \{0, 1, \dots, s-1\}$.

Remark 2. Every cylindrical interval contains s pairwise disjoint cylindrical intervals of the next rank.

For a cylindrical interval we introduce the notation

$$t_j := \frac{\alpha_1}{s} + \dots + \frac{\alpha_n}{s^n} - \frac{M-1-j}{s^n}, \quad j \in \{0, 1, \dots, M-1\},$$

where $M = [\frac{r}{s}] + 1$. Thus t_{M-1} is the left endpoint of the cylindrical interval, and the points t_j are obtained by shifting this point to the left by $\frac{M-1-j}{s^n}$.

Definition 3. The *prefix of a cylindrical interval* of rank n is the ordered tuple $(p_0, p_1, \dots, p_{M-1})$ of zeros and ones, where $p_i = 1$ if $t_i \in \mathcal{E}(\sum_{k=1}^{n} \xi_k s^{-k})$, and $p_i = 0$ otherwise.

Definition 4. The *prefix graph* G' is the graph defined as follows:

1) its vertices are all possible prefixes of cylindrical intervals that can be reached from the vertices of the form

$$(1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1);$$

2) for every vertex, s directed edges are defined, going from it to those vertices that are prefixes of the cylindrical intervals of the next rank contained in the given cylindrical interval.

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Definition 5. A partial equivalence class is called *terminal* if no vertex of the class has a directed outgoing edge leading to a vertex of another class or to a transient vertex.

Theorem 6. *The set $\mathcal{E}(\xi)$ of values of a random power series contains an interval if and only if the corresponding directed prefix graph has a strongly connected component different from $\{(0,0,\ldots,0)\}$ from which there is no outgoing edge to another strongly connected component of the graph.*

Definition 7. The *colors of partial equivalence classes and transient vertices* are defined by the following rules:

The terminal class $\{(0,0,\ldots,0)\}$ is assigned the *white* color, while all other terminal classes are assigned the *black* color. Then, iteratively, each class (or transient vertex) is assigned a color depending on the colors of its immediate successors, namely:

- *white*, if all successors are white;
- *black*, if all successors are black;
- *black-and-white*, if there are successors of both colors, or if there is at least one black-and-white successor.

Lemma 8. *If the condensed graph $\bar{G}'$ contains a non-cyclic black-and-white class, then the set $\mathcal{E}(\xi)$ contains infinitely many intervals.*

Lemma 9. *If for two distinct black-and-white classes there exists a directed path from one of them to the other, then the set $\mathcal{E}(\xi)$ contains infinitely many intervals.*

Let an edge $v \rightarrowtail w$ of the graph G' correspond to the transition from a cylindrical interval $\widetilde{\Delta}_{\alpha_1\ldots\alpha_n}$ to its subcylinder $\widetilde{\Delta}_{\alpha_1\ldots\alpha_n d}$, $(d \in \{0,1,\ldots,s-1\})$. The number d will be called the *index* of the edge and denoted by $\operatorname{ind}(v,w)$.

Let v be a vertex of some cyclic partial equivalence class C, and let k be the index of the unique edge that goes within the class to a vertex $v_2 \in C$. For v, we split the set of its other immediate successors into two sets

$$S^-(v) := \{\, w \mid v \rightarrowtail w,\ \operatorname{ind}(v,w) < k\},$$

$$S^+(v) := \{\, w \mid v \rightarrowtail w,\ \operatorname{ind}(v,w) > k\}.$$

Definition 10. We say that a vertex v of a cyclic class has *property D_1* if all vertices in $S^-(v)$ are black and all vertices in $S^+(v)$ are white. Analogously, the vertex v has *property D_2* if all vertices in $S^-(v)$ are white and all vertices in $S^+(v)$ are black.

Lemma 11. *For the set $\mathcal{E}(\xi)$ to contain only finitely many intervals it is necessary that, in every black-and-white partial equivalence class, all vertices have the same property — either all have D_1, or all have D_2.*

Theorem 12. Criterion for the finiteness of the number of intervals in $\mathcal{E}(\xi)$. *The set $\mathcal{E}(\xi)$ contains only finitely many intervals if and only if the following conditions are simultaneously satisfied:*

- *1) all black-and-white classes are cyclic;*
- *2) there are no directed paths between distinct black-and-white classes;*
- *3) for every black-and-white class, all of its vertices have the same property — either D_1 or D_2.*

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Geometry vs. Topology: Classifying Curves via Zariski Pairs

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The interplay between geometry and topology is one of the most fascinating aspects of algebraic geometry. While two plane algebraic curves can look identical from a local or combinatorial perspective, their global embeddings into the projective plane can behave in entirely different ways.

The Zariski pairs serve as the perfect manifestation of this phenomenon, showing exactly where the pure algebraic data diverge from the topological reality.

The theory originated from this classical observation of Oscar Zariski—that two algebraic curves may share the same combinatorics and local singularity structure, while having non-homeomorphic embeddings in the projective plane.

Since the 1990s, the subject has developed into a rich interaction between algebraic geometry, topology, singularity theory, braid groups, arithmetic geometry, and computational methods, serving as a powerful framework for the classification of algebraic curves. A major turning point occurred with the work of Enrique Artal Bartolo in ”Sur les couples de Zariski” [4]. In this paper, Bartolo systematically constructed Zariski pairs (using Alexander polynomials) and showed that combinatorics alone does not determine the topology of the complement of a plane curve. This work established a modern framework for studying the embedded topology of algebraic curves through topological invariants. This direction was expanded in Zariski pairs, fundamental groups and Alexander polynomials [7], where the topology of complements and fundamental groups became central tools. The paper develops techniques for distinguishing sextic curves with identical singularities and strongly influenced later work on line arrangements.

Ichiro Shimada introduced lattice-theoretic methods in ”A note on Zariski pairs” [16]. This work connects Zariski pairs with K3 surfaces and brings arithmetic and algebraic-geometric methods into the subject.

The year 2000 brought further deep arithmetic-geometric perspectives into the field. An example of such progress can be found in [13], where arithmetic properties of elliptic K3 surfaces were related directly to embedded topology.

A new type of sextic Zariski pair was presented by Artal Bartolo, Carmona, Cogolludo, and Tokunaga in ”Sextics with singular points in special position” [11], showcasing curves with triple points lying in special positions.

In the early 2000s, braid monodromy became one of the strongest invariants in the field. In ”Braid monodromy and topology of plane curves” [8], braid monodromy was shown to encode refined topological information capable of distinguishing curves with identical combinatorics. This approach was further developed in ”Effective invariants of braid monodromy” [9], where computable

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braid monodromy invariants were introduced. At the same time, work on arrangements explicitly demonstrated that combinatorics does not determine topology.

In ”Topology and combinatorics of real line arrangements” [10], the authors constructed arrangements with identical incidence structures but different embeddings in the projective plane. This result became fundamental for the study of Zariski pairs of line arrangements.

During the 2010s, several new invariants appeared in ”A linking invariant for algebraic curves” [15], ”An arithmetic Zariski pair of line arrangements with non-isomorphic fundamental group” [12], and ”Torsion divisors of plane curves with maximal flexes and Zariski pairs” [6]. Recent research shifted toward realization spaces, conic-line arrangements, and computational methods. In Zariski pairs of conic-line arrangements of degrees 7 and 8 via fundamental groups [3], new Zariski pairs were constructed using van Kampen techniques and Coxeter groups. Later, in ”The realization space of a certain conic-line arrangement of degree 7 and a π_1-equivalent Zariski pair” [1], the authors showed that even arrangements with identical fundamental groups may still form Zariski pairs. This demonstrated that π_1 itself is not always sufficient for classification. There is also an investigation of Tokunaga's arrangements. One of them is shown in the following figure.

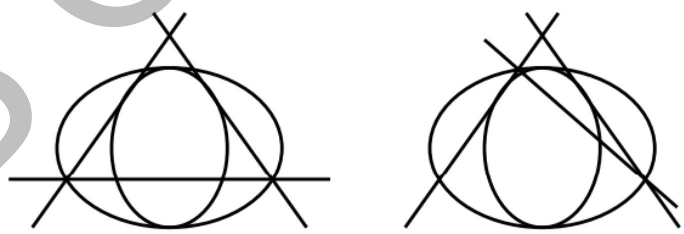

Figure 0.1. Tokunaga's arrangement

The modern viewpoint is summarized also in ”Topology of complex plane curves: braid monodromy, local and global problems” [5], where braid monodromy is presented as a unifying framework connecting singularity theory, topology of complements, arithmetic geometry, realization spaces, and classification problems.

In my talk, I will merge all the above findings and background into a deep discussion that also includes fundamental group computations and software construction and utilization. Our recent work is summarized in ”Detecting Zariski pairs by algorithms and computational classification in conic line arrangements” [2], where algorithmic and combinatorial techniques are used to classify arrangements and detect candidate Zariski pairs systematically. Motivated by known examples of Zariski pairs featuring one conic and 7 lines (in [14]), our findings establish a systematic inductive algorithm to classify $(n, 1)$-arrangements. In the talk, I will discuss how this approach proves that no Zariski pairs exist for $n \leq 4$. We will also present our newly constructed $(2, 3)$-arrangements, demonstrating that even curves with isomorphic fundamental groups can form Zariski pairs. I will talk on further plans to optimize these computational methods to handle a larger number of lines and to extend our combinatorial equivalence framework to support multiple conics. An example of a Zariski pair from [14] is depicted in the following figure (one conic and 7 lines).

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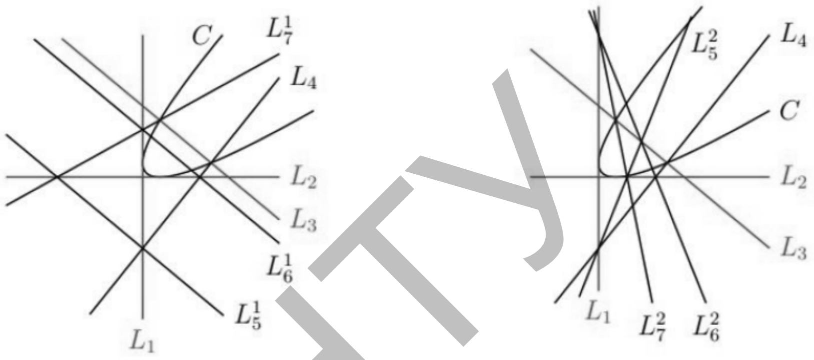

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Horizontal Gauss maps of hypersurfaces in 2-step Carnot groups

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Let G be a $(n+1)$-dimensional Lie group with a left invariant metric $\langle\cdot,\cdot\rangle$. For an (immersed) oriented hypersurface M in G with the unit normal field η define the *(left-invariant) Gauss map*

$\Phi\colon M \to S^n$ of M by $\Phi(p) = dL_{p^{-1}}(\eta(p))$, where a point $p \in M$ is identified with its image under the immersion, $dL_{p^{-1}}$ is the differential of the left translation in G, and S^n is the unit sphere in the Lie algebra $\mathfrak{g}$ of G with respect to $\langle\cdot,\cdot\rangle$. This is a generalization of the Gauss map of a hypersurface in the Euclidean space, that is harmonic if and only if M is of constant mean curvature (CMC) by a well-known result of E.A. Ruh and J. Vilms ([5]). This equivalence does not take place in general. For example, each CMC hypersurface with the harmonic Gauss map in the $(2m+1)$-dimensional Heisenberg group is locally a vertical cylinder ([4]). Note that for a Riemannian manifold M a map $\Phi\colon M \to S^n$ is harmonic if and only if $\Delta\Phi = 2e(\Phi)\,\Phi$, where Δ is the Riemannian Laplacian of M, $\Delta\Phi = (\Delta\Phi^1, \dots, \Delta\Phi^{n+1})$ for $\Phi = (\Phi^1, \dots, \Phi^{n+1})$, and $e(\Phi)$ is the energy density function of Φ.

Let now G be a 2-*step nilpotent group*. This means that for the (non-trivial) center $\mathfrak{z}$ of $\mathfrak{g}$ and the orthogonal complement $\mathfrak{h}$ of $\mathfrak{z}$ with respect to $\langle\cdot,\cdot\rangle$ we have $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{z}$. Assume additionally that $[\mathfrak{h},\mathfrak{h}] = \mathfrak{z}$, i.e., G is a 2-*step Carnot group*. Then the left-invariant distribution $\mathcal{H}$ generated by $\mathfrak{h}$ is totally non-integrable and, together with the restriction of $\langle\cdot,\cdot\rangle$ to $\mathcal{H}$, defines a sub-Riemannian structure on G. For any $p \in G$ let us call elements of $\mathcal{H}_p$ *horizontal*, and elements of $\mathcal{Z}_p$, where $\mathcal{Z}$ is the left-invariant distribution generated by $\mathfrak{z}$, *vertical*.

Let a neighborhood $U \subset M$ consist of non-singular points, i.e. $\mathcal{H}_p \not\subset T_pM$ for any $p \in U$. Then the intersections $\mathcal{H}_p \cap T_pM$ form a q-dimensional distribution on U, where $\dim\mathfrak{h} = q+1 < n+1$. Together with the restriction of the first fundamental form of M, it yields a sub-Riemannian structure on U. Let Δ_H be a sub-Riemannian Laplacian of this structure ([1]). For each $p \in U$ the orthogonal projection $\eta(p)^{\mathcal{H}}$ of $\eta(p)$ to $\mathcal{H}_p$ is non-zero, so the *horizontal unit normal field* $\eta_H\colon p \mapsto \eta(p)^{\mathcal{H}}/|\eta(p)^{\mathcal{H}}|$ is well defined on U. Similarly to the previous definition of the Gauss map, we can then call

$$\Phi_H\colon U \to S^q\colon p \mapsto dL_{p^{-1}}(\eta_H(p))$$

the *(left-invariant) horizontal Gauss map* to the unit sphere of $\mathfrak{h}$.

The *horizontal second fundamental form* b_H on U is constructed as follows ([3]). For any two horizontal tangent vector fields X, Y on U, i.e., $X(p), Y(p) \in \mathcal{H}_p \cap T_pM$ for any $p \in U$, $b_H(X,Y) = \langle\nabla_X Y, \eta_H\rangle$, where ∇ is the Riemannian connection of $\langle\cdot,\cdot\rangle$. Let $\{E_k\}_{k=1}^q$ be a local orthonormal frame of the distribution $p \mapsto \mathcal{H}_p \cap T_pM$. Then $\sum\limits_{k=1}^{q} b_H(E_k, E_k)$, i.e., the trace of b_H with respect to $\langle\cdot,\cdot\rangle$, is called the *horizontal mean curvature* of U, and $|b_H|^2 = \sum\limits_{k,l=1}^{q} b_H(E_k, E_l)^2$ is the *square norm* of b_H of U with respect to $\langle\cdot,\cdot\rangle$.

Theorem 1. *Let p be a non-singular point of M and $U \subset M$ be a neighborhood of p consisting of non-singular points. Let $\eta(p) = \lambda X_{q+1} + \mu Z_{n-q}$, where X_{q+1} and Z_{n-q} are unit horizontal and vertical vectors, respectively, and let $\{X_k\}_{k=1}^q$ be an orthonormal basis of $\mathcal{H}_p \cap T_pM$. Denote also by X_k left-invariant vector fields extending these vectors and corresponding to elements of $\mathfrak{h}$. Then*

$$\Delta_H\Phi_H(p) = \sum_{k=1}^{q}\Big(- X_k(H) + \mathrm{Ric}\,(X_k, X_{q+1})\Big)\,X_k - \Big(|b_H|^2(p) + \frac{1}{2}\,\mathrm{Ric}\,(X_{q+1}, X_{q+1})\Big)\,X_{q+1},$$

where H is the horizontal mean curvature of U and Ric *is the Ricci tensor of $\langle\cdot,\cdot\rangle$.*

Note that $\Phi_H(p) = X_{q+1}$. Similarly to the harmonicity criterion for maps into spheres stated above, we will call Φ_H *horizontally harmonic* at p, if $\Delta_H\Phi_H(p)$ is proportional to X_{q+1}. For each $Z \in \mathfrak{z}$ denote by $J(Z)\colon \mathfrak{h} \to \mathfrak{h}$ the linear operator defined by $\langle J(Z)X, Y\rangle = \langle [X,Y], Z\rangle$ for any $X, Y \in \mathfrak{h}$. It is said that G is *of Heisenberg type* ([2]), if $J(Z)^2 = -|Z|^2 Id_{\mathfrak{h}}$. The $(2m+1)$-dimensional Heisenberg group is an example of such a group. For a group G of Heisenberg type and any $X, Y \in \mathfrak{h}$ we have $\operatorname{Ric}(X, Y) = -\frac{1}{2}(n-q)\langle X, Y\rangle$, therefore Theorem 1 implies the following analogue of the Ruh–Vilms theorem:

Corollary 2. *Let G be of Heisenberg type. Then the horizontal mean curvature of U is constant in all horizontal directions if and only if the horizontal Gauss map of U is horizontally harmonic.*

We will also discuss possible applications of these results.

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Symmetry properties of symmetric Nyzhnyk models

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Applying the inverse scattering method to integrating multidimensional nonlinear equations in [2], Leonid Nyzhnyk constructed one of the first multidimensional integrable models

$$w_t = k_1 w_{xxx} + k_2 w_{yyy} + 3(v^1 w)_x + 3(v^2 w)_y, \quad v^1_y = k_1 w_x, \quad v^2_x = k_2 w_y, \tag{1}$$

which is now called the *Nyzhnyk system*. Since k_1 and k_2 are arbitrary constant parameters subject to the condition $(k_1, k_2) \neq (0, 0)$, the system (1) in fact represents a class of systems, which is the union of two cosets with respect to scaling equivalence transformations and the permutation of (x, v^1) and (y, v^2). Namely, in the *symmetric* case, where both parameters k_1 and k_2 are nonzero, $k_1 k_2 \neq 0$, one can set $(k_1, k_2) = (1, 1)$, whereas in the *asymmetric* case, where exactly one of them vanishes, one can set $(k_1, k_2) = (1, 0)$. Using the introduction of potentials, limiting procedures, interpretations of independent and/or dependent variables as real or complex, and differential substitutions, from the system (1) one can derive a considerable number of various integrable models, which we refer to as *Nyzhnyk models* [3].

Among these models, one distinguishes symmetric and asymmetric, dispersive and dispersionless, standard models with quadratic nonlinearities and modified models with cubic nonlinearities, single partial differential equations and systems of such equations, real, complex, mixed, and specific models, where complex variables are split into pairs of complex-conjugate variables, as well as

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linear and nonlinear nonisospectral Lax representations in the dispersive and dispersionless cases, respectively. According to the sign of nonlinearities, modified models are additionally divided into defocusing and focusing ones. There also exist generalizations of the Nyzhnyk models with larger numbers of independent or dependent variables. The classification of Nyzhnyk models presented in [3] makes it possible to clearly identify structural and functional distinctions between groups of these models, thereby creating a basis for their systematic analysis as well as for revealing their genetic relations with other equations of mathematical physics.

Each of the Nyzhnyk models possesses interesting symmetry properties but they are not well studied. Thus, the extended classical symmetry analysis has been carried out completely only for the (real symmetric potential) dispersionless Nyzhnyk equation [1, 4, 5]. At the same time, relations between the maximal Lie invariance pseudoalgebras of Nyzhnyk models according to various relations between these models have been revealed. In particular, it turned out that the maximal Lie invariance pseudoalgebra of each dispersive Nyzhnyk model is a codimension-one pseudosubalgebra of the maximal Lie invariance pseudoalgebra of its dispersionless counterpart, where the complementary part is associated with scaling transformations of the spatial variables. Moreover, in order to derive the maximal Lie invariance pseudoalgebra of a Lax representation of a Nyzhnyk model, it suffices to prolong the elements of the pseudoalgebra of this model to the pseudopotential regarded as a new dependent variable and to add one more generating vector field associated with the arbitrariness in the definition of the pseudopotential.

There exists a relation between the point-symmetry pseudogroups of the corresponding dispersionless and dispersive Nyzhnyk models that is similar to that between their maximal Lie invariance pseudoalgebras. As an example, consider the (real symmetric potential) dispersive and dispersionless Nyzhnyk equations

$$u_{txy} = u_{xxxxy} + u_{xyyyy} + (u_{xx}u_{xy})_x + (u_{yy}u_{xy})_y, \tag{2}$$

$$u_{txy} = (u_{xx}u_{xy})_x + (u_{yy}u_{xy})_y. \tag{3}$$

The maximal Lie invariance pseudoalgebras of the equations (2) and (3) are the (infinite-dimensional) pseudosubalgebras

$$\mathfrak{g}_2 := \langle D^t(\tau),\, P^x(\chi),\, P^y(\rho),\, R^x(\alpha),\, R^y(\beta),\, Z(\sigma)\rangle, \quad \mathfrak{g}_3 := \mathfrak{g}_2 + \langle D^{\rm s}\rangle,$$

respectively, where

$$D^t(\tau) = \tau\partial_t + \tfrac13\tau_t x\partial_x + \tfrac13\tau_t y\partial_y - \tfrac1{18}\tau_{tt}(x^3+y^3)\partial_u, \quad D^{\rm s} = x\partial_x + y\partial_y + 3u\partial_u,$$

$$P^x(\chi) = \chi\partial_x - \tfrac12\chi_t x^2\partial_u, \quad P^y(\rho) = \rho\partial_y - \tfrac12\rho_t y^2\partial_u, \quad R^x(\alpha) = \alpha x\partial_u, \quad R^y(\beta) = \beta y\partial_u, \quad Z(\sigma) = \sigma\partial_u.$$

Thus, the pseudoalgebra $\mathfrak{g}_2$ is a codimension-one pseudosubalgebra of the pseudoalgebra $\mathfrak{g}_3$.

Theorem 1. *(i) The point-symmetry pseudogroup G_3 of the dispersionless Nyzhnyk equation (3) is generated by the transformation $\mathscr{J}\colon \tilde t = t$, $\tilde x = y$, $\tilde y = x$, $\tilde u = u$ and the transformations of the form*

$$\begin{aligned}
&\tilde t = T(t), \quad \tilde x = CT_t^{1/3}x + X^0(t), \quad \tilde y = CT_t^{1/3}y + Y^0(t),\\
&\tilde u = C^3u - \frac{C^3T_{tt}}{18T_t}(x^3+y^3) - \frac{C^2}{2T_t^{1/3}}(X^0_t x^2 + Y^0_t y^2) + W^1(t)x + W^2(t)y + W^0(t),
\end{aligned} \tag{4}$$

where T, X^0, Y^0, W^0, W^1 and W^2 are arbitrary smooth functions of t with $T_t \neq 0$, and C is an arbitrary nonzero constant.

(ii) The contact-symmetry pseudogroup $G_3^{\rm c}$ of the dispersionless Nyzhnyk equation (3) coincides with the first prolongation of the pseudogroup G_3.

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(iii) The point- and contact-symmetry pseudogroups G_2 and G_2^{c} of the dispersive Nyzhnyk equation (2) *is a pseudosubgroup of the pseudogroups G_3 and G_3^{c} that are singled out by the constraint $C = 1$.*

This work was supported in part by grants from the Simons Foundation (SFI-PD-Ukraine-00014586, O.O.V., V.M.B.). O.O.V. acknowledges the financial support of the National Academy of Sciences of Ukraine within the framework of the project 0125U002856 for young scientists and extends her sincere thanks to the Mathematical Institute of the Silesian University in Opava for their hospitality and support during her research visits. R.O.P. expresses his gratitude for the hospitality shown by the University of Vienna during his long-term stay there. The work of R.O.P. was supported in part by the Ministry of Education, Youth and Sports of the Czech Republic (MŠMT ČR) under RVO funding for IČ47813059.

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Comaximal Graphs of commutative ring, Edge Ideals, Betti Numbers, Hilbert Series, and Homotopy Type

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We study the comaximal graphs of the finite rings $\mathbb{Z}_n$ and explains how their arithmetic structure influences algebraic and topological invariants. The vertices are partitioned into units and nonunits, and this decomposition yields a precise description of adjacency in terms of gcd-classes. Using this structure, the edge ideal of $\Gamma(\mathbb{Z}_n)$ is analyzed through the language of Stanley–Reisner theory and independence complexes. Hochster's formula is then applied to derive information about the graded Betti numbers, especially the linear strand. In important cases, explicit formulas for $\beta_{i,i+1}$ are obtained, and the extremal Betti number is shown to satisfy $\beta_{n-1,n} = \varphi(n)$. Also, we highlight the consequences that $\operatorname{reg}(I(\Gamma(\mathbb{Z}_n))) = 2$ and $\operatorname{pdim}(S/I) = n - 1$. On the enumerative side, the Hilbert series is computed from the independence polynomial, f-vector, and h-polynomial of the associated simplicial complex. A topological interpretation is given via the independence complex, whose homotopy type is a wedge of $\varphi(n)$ copies of circles S^0. This decomposition shows that the nonunit part is contractible, while the units contribute isolated components. As a consequence, the edge ring is Cohen–Macaulay precisely when n is prime.

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On non-existence of continuous decomposition of symmetric matrices into a linear combination of orthoprojectors

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Recall that a linear operator $P\colon \mathbb{R}^n \to \mathbb{R}^n$ is
- *symmetric* (selfadjoint) if $P^t = P$;
- *orthoprojector* if $P^t = P$ and $P^2 = P$.

For $k \geq 1$ denote by
- $\mathcal{P}^n_k$ be the set of orthoprojectors onto subspaces of dimension k, i.e. having rank k;
- $\mathcal{P}^n = \cup_{i=0}^{n} \mathcal{P}^n_k$ be the set of all orthoprojectors;
- Sym^n be the set of all symmetric $n \times n$-matrices.

Clearly, $\mathcal{P}^n \subset Sym^n$. Moreover, $\mathcal{P}^n_k$ is naturally identified with the corresponding Grassmanian manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$.

There is an extensive literature discussing the problem of representing symmetric operators as linear combinations of orthoprojectors, i.e. for given a matrix $A \in Sym^n$ one looks for some numbers $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ and orthoprojectors $P_1, \dots, P_m \in \mathcal{P}^n$ such that

$$A = \lambda_1 P_1 + \dots + \lambda_k P_k.$$

This problem has a lot of variations, say one can put a restriction on the number k of orthoprojectors and their ranges.

It can be reformulated as follows. Consider the map

$$\phi_k\colon \mathbb{R}^k \times (\mathcal{P}^n)^k \to Sym^n, \qquad \phi_k(\lambda_1, \dots, \lambda_k, P_1, \dots, P_k) = \lambda_1 P_1 + \dots + \lambda_k P_k.$$

Then the question is to describe the image of this map.

Since every $A \in Sym^n$ has an orthonormal basis consisting of eigen vectors (spectral decomposition theorem), the map ϕ_k is surjective for $k \geq n$.

Note also that associating to a matrix A some $2n$-tuple $(\lambda_1, \dots, \lambda_k, P_1, \dots, P_k)$ such that $A = \phi_k(\lambda_1, \dots, \lambda_k, P_1, \dots, P_k)$ can be regarded as a *section* of ϕ_k, i.e. a map

$$s\colon Sym^n \to \mathbb{R}^k \times (\mathcal{P}^n)^k$$

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such that $\phi_k \circ s = \mathrm{id}_{Sym^n}$. It is easy to see that ϕ_k admits a section if and only if it is surjective. However, such a section is not always *continuous*.

The aim of the talk is to discuss existence of *continuous* sections of ϕ_k and obstructions to constructing them. In particular we will prove the following

Theorem 1. *The map*

$$\phi_2 \colon \mathbb{R}^2 \times \mathcal{P}_1^2 \times \mathcal{P}_1^2 \to Sym^2$$

is surjective, but it does not admit a continuous section.

In other words, it is not possible to associate to each symmetrix 2×2 matrix $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$ a pair of coefficients $\lambda_1(A), \lambda_2(A) \in \mathbb{R}$, and a pair of orthoprojectors $P_1(A), P_2(A) \in \mathcal{P}_1^2$ onto some lines in $\mathbb{R}^2$ so that

$$A = \lambda_1(A)P_1(A) + \lambda_2(A)P_2(A),$$

and these functions $\lambda_1, \lambda_2, P_1, P_2$ will be continuous in the coefficients of A.

The proof is based on the non-triviality of the second homotopy group $\pi_2(S^2) = \mathbb{Z}$ of 2-sphere.

Representations of the p-adic rotation group towards p-adic quantum computing

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We investigate the structure of the p-adic rotation group $SO(3)_p$, and initiate a program aimed at classifying its finite-dimensional irreducible projective unitary representations. Our approach relies on the profinite nature of $SO(3)_p$, together with its Haar measure. These representations can be interpreted as a theory of p-adic angular momentum and spin, where the p-adic qubit arises as a two-dimensional representation. Indeed, p-adic numbers find fruitful applications in p-adic formulations of quantum mechanics, and in dynamical systems, where their ultrametric topology naturally models hierarchical and fractal-like structures.

We describe the foundations of our program, starting from the main features of $SO(3)_p$ (in parallel to its real counterpart), such as a p-adic analogue of the Cardano (aka nautical) angles decomposition [1], and a descriprion of rotations in terms of p-adic quaternions [3]. We characterise the profinite group $SO(3)_p$ as an inverse limit of the inverse family of groups $SO(3)_p \bmod p^k$ [4]. We show that all finite-dimensional projective unitary representations of $SO(3)_p$ factorise on some $SO(3)_p \bmod p^k$, and we find explicit p-adic qubit representations for every prime p. Interestingly, there are several p-adic qubit representations for $p > 3$ [2].

Then, it is natural to compose systems of multiple p-adic qubits, through the tensor product of their representations. We solve the Clebsch-Gordan problem for systems of two p-adic qubits from $SO(3)_p \bmod p$, revealing that the coupled bases decompose into singlet and doublet states [6]. We further study entanglement arising from those stable subsystems: every singlet, doublet (and triplet) can be given by maximally entangled Bell states. However, except for the singlets, the projectors onto doublets (and triplets) are separable quantum states [5].

Lastly, we propose a circuit model of quantum computation where logic gates are driven by the actions of $SO(3)_p$ [6]. For $p = 3$, we construct a set of gates from four-dimensional irreducible representations of $SO(3)_3 \bmod 3$, that we prove to be universal for quantum computation [5].

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Inversor of digits in the B-representation of numbers on the unit interval and its automodel and integro-differential properties

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Let $A = Z = \{0, \pm 1, \pm 2, ...\}$ be an alphabet (a set of digits), $L = A \times A \times ...$ — the set (space) of sequences of elements of the alphabet. Let (Θ_n) be a two-sided sequence of positive real numbers $(n \in Z)$ such that

$$0 < \sum_{n=1}^{\infty} \Theta_{-n} \equiv u < 1, \; 0 < \sum_{n=0}^{+\infty} \Theta_n \equiv v < 1, \; u + v = 1;$$

(b_n) – is another two-sided sequence defined by the sequence (Θ_n), namely

$$b_n \equiv \sum_{i=-\infty}^{n-1} \Theta_i = b_{n-1} + \Theta_{n-1}.$$

Theorem 1. *For any number $x \in (0; 1)$ there exists a unique finite set of integers $(\alpha_1, \alpha_2, ..., \alpha_m)$ or a unique sequence $(\alpha_n) \in L$ such that one of the following equalities holds*

$$x = b_{\alpha_1} + \sum_{k=2}^{m} b_{\alpha_k} \prod_{i=1}^{k-1} \Theta_{\alpha_i} \equiv \Delta^B_{\alpha_1 \alpha_2 ... \alpha_m (\emptyset)} \tag{1}$$

$$x = b_{\alpha_1} + \sum_{k=2}^{\infty} b_{\alpha_k} \prod_{i=1}^{k-1} \Theta_{\alpha_i} \equiv \Delta^B_{\alpha_1 \alpha_2 ... \alpha_k ...} \tag{2}$$

Symbolic representation of a number x by equality (1) or (2) is called the *B-representation* of this number, and $\alpha_n = \alpha_n(x)$ – is its nth digit. Due to the uniqueness of the B-representation, the value $\alpha_n = \alpha_n(x)$ is a correctly defined function of the number x. Let us note that the B-representation of numbers has extra-zero redundancy (each number possesses a unique B-representation).

Definition 2. The *inversor of digits* in the B-representation of numbers is defined as the function I, given by the equalities:

$$I(\Delta^B_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \Delta^B_{[-\alpha_1][-\alpha_2]\ldots[-\alpha_n]\ldots},\quad I(\Delta^B_{\alpha_1\alpha_2\ldots\alpha_n(\emptyset)}) = \Delta^B_{[-\alpha_1][-\alpha_2]\ldots[-\alpha_n](\emptyset)}.$$

Lemma 3. *The inversor I of digits in the B-representation of numbers has the following properties:*

1) $I(\Delta^B_{(0)}) = \frac{b_0}{1-\Theta_0} = \frac{u}{1-\Theta_0}$;

2) is a continuous and strictly decreasing function.

Theorem 4. *The graph Γ_I of the inversor I is an N-self-affine set, that is,*

$$\Gamma_I = \bigcup_{i\in\mathbb{Z}} f_i(\Gamma_I),$$

where the affine mappings f_i have the form:

$$f_i : \begin{cases} x' = \Theta_i x + b_i, \\ y' = \Theta_{-i} y + b_{-i}. \end{cases}$$

The self-affine dimension of the graph of the inverter is the solution of the equation

$$\sum_{n=-\infty}^{+\infty} \begin{vmatrix} \Theta_n & 0 \\ 0 & \Theta_{-n} \end{vmatrix}^{\frac{x}{2}} = 1. \tag{3}$$

In particular, when $\Theta_n = \Theta_{-n}$, then equation (3) takes the form

$$\sum_{n=-\infty}^{\infty} \Theta_n^x = 1.$$

When $\sum_{n=0}^{\infty} \Theta_n^x = 1$, then equation (3) takes the form

$$\frac{1}{2^x} + \sum_{n=1}^{\infty} \frac{1}{2^{nx}} = 1 \Leftrightarrow \frac{1}{2^x} + \frac{1}{2^x - 1} = 1.$$

When $\Theta_n = \Theta_{-n} = \frac{1}{2^{n+2}}$ for $n \geq 1$ and $\Theta_0 = \frac{1}{2}$, then equation (3) takes the form

$$\Theta_0^x + 2\sum_{n=1}^{\infty} \Theta_n^x = 1.$$

Theorem 5. *If there is such n, that $\Theta_n \neq \Theta_{-n}$, then the inverter is a singular function (a function which is continuous, non-constant, and has a derivative equal to zero almost everywhere in the sense of Lebesgue measure) and when $\Theta_n = \Theta_{-n}$ for all $n \in Z$ the equation holds $I(x) = -x$.*

Theorem 6. *For the inversor $I(x)$ on the interval $[0,1]$ the following equality holds:*

$$\int_0^1 I(x)\,dx = \frac{\sum\limits_{i\in\mathbb{Z}} b_i\Theta_i}{1 - \sum\limits_{i\in\mathbb{Z}} \Theta_i^2}.$$

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Sub-Hamiltonian Framework for Sub-Finsler Geometry

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A Hamiltonian space is a pair $(M, \tilde{H})$, where M is an n-dimensional manifold and $\tilde{H} : T^*M \to \mathbb{R}$ is a smooth Hamiltonian function with nondegenerate Hessian determinant. It was introduced and studied by R. Miron [5]. Hamiltonian geometry is the branch of differential geometry that studies Hamiltonian spaces, Hamiltonian systems, symplectic structures, canonical flows, connections, curvature, Hamiltonian vector fields, and related geometric-dynamical properties. It plays a fundamental role in various areas of mathematics and physics. The works of R. Miron and his co-authors have made significant contributions to the development and advancement of Hamiltonian geometry, providing valuable insights into the geometric foundations of Hamiltonian systems, [6]. Hamiltonian geometry provides a natural framework for studying geodesics, optimal control, and dynamical structures in sub-Riemannian and sub-Finsler geometry, particularly through Hamiltonian flows and Legendre duality, see [1, 7] for more details.

We develop sub-Finsler bundle geometry and its relationship with sub-Hamiltonian functions on a sub-Finsler manifold $(M, \mathcal{D}, F)$. The sub-Hamiltonian $H : \mathcal{D}^* \to \mathbb{R}$ is positive semi-definite, vanishing only at $p = 0$. Also, we establish a three-way equivalence characterizing forward geodesic completeness via global sub-Hamiltonian flows and compactness of forward distance balls, as a consequence of the sub-Finsler Hopf–Rinow theorem. Moreover, under completeness and controllability, existence of a time-optimal control is proved through properness of the sub-Finsler distance. The Legendre transformation reduces energy minimization to a first-order ODE on $\mathcal{D}^*$, yielding normal geodesics as solutions. Furthermore, three dynamical results follow: geodesics are integral curves of $\vec{H}$; the Liouville volume form on T^*M is preserved under the flow of $\vec{H}$ via canonical extension of H; and $\vec{H}$ is symplectically orthogonal to all $\mathcal{D}$-preserving vector fields in $\mathcal{D}_x^0$, encoding nonholonomic constraints geometrically. These results establish a rigorous Hamiltonian framework for sub-Finsler geometry with direct applications to nonholonomic mechanics, robotics, and optimal control. Lastly, abnormal extremals and three open problems are identified.

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Invariance in time of α-convexity of order β

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The Hele–Shaw problem models the motion of a viscous fluid contained between two closely spaced parallel plates. Its evolution is described by a free boundary whose dynamics can be represented through conformal mappings satisfying the Polubarinova–Galin equation. This equation provides a powerful analytic framework linking geometric function theory to free-boundary dynamics in planar domains.

In this paper, we focus on the hereditary properties of α-convex functions of order β under the assumptions of the Hele–Shaw model. These functions form a two-parameter family that unifies and extends the classical starlike and convex classes. We analyze how the α-convexity of order β behaves under the conformal evolution dictated by the Hele–Shaw equation and identify sufficient conditions for which this property is preserved in time. The results establish a direct connection between analytic convexity and geometric stability, revealing how structural properties from univalent function theory control the invariance of evolving domains in Hele–Shaw-type flows.

In this paper we obtain a result which proves that the α- convexity of order β is preserved in time. The result is a generalisation of Theorem 2.3 form [1] on the case of α-convexity of order β. We can obtain Theorem 2.3 by taking $\beta = 0$.

Theorem 1. *Let $Q < 0$ and f_0 an univalent function on $\overline{U}$. Then, there exists $\alpha_0 \in (0;1)$ and $\beta \in [0;1)$ such that, if f is an α-convex of order β function on U, $\alpha < \alpha_0$, then the classical solution of the Polubarinova Galin equation with the initial condition $f(z;0) = f_0(z)$ remains $\alpha-$ convex of order β during the existence time. Moreover, the existence time of the classical solution is infinite.*

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Hamiltonian Lax type flows related to the centrally extended Lie algebra of matrix super-integro-differential operators with two anticommuting variables and their rational factorization

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In the paper [1] the Lie algebra of matrix super-integro-differential operators with one anticommuting variable, which possesses an invariant with respect to the corresponding commutator scalar product and standard splitting into a direct sum of its Lie subalgebras, has been considered. It has been applied to construct Lax integrable hierarchies of nonlinear dynamical systems on supermatrix-valued supermanifolds of one commuting and one anticommuting independent variables as reductions of the hierarchy of Hamiltonian Lax type flows in its dual space, generated by the associated $\mathcal{R}$-deformed Lie-Poisson bracket and Casimir functionals, upon coadjoint orbits. The mentioned hierarchy and set of hierarchies of its additional Hamiltonian homogeneous symmetries for certain type coadjoint orbits have been used to obtain the nonlinear dynamical systems on supermatrix-valued supermanifolds of two commuting and one anticommuting independent variables, which admit triple Lax type linearizations.

Basing on the Lie-algebraic Adler-Kostant-Symes scheme and $\mathcal{R}$-operator approach, in [2] one obtains the hierarchy of Hamiltonian Lax type flows in the dual space to the centrally extended by the Maurer-Cartan two-cocycle parameterized Lie algebra of matrix super-integro-differential operators with one anticommuting variable. On the corresponding coadjoint orbits it is reduced to some Lax integrable hierarchies of nonlinear dynamical systems on supermatrix-valued supermanifolds of two commuting and one anticommuting independent variables. There the rational factorization (see, for example, [3]–[6]) of this hierarchy has been also investigated.

In the report, that is proposed, one introduces the Lie algebra of matrix super-integro-differential operators with two anticommuting variables, endowed with an invariant scalar product, which admit a non-standard splitting into a a direct sum of its Lie subalgebras and develops a new Lie-algebraic method for constructing Lax integrable hierarchies of nonlinear dynamical systems on supermatrix-valued supermanifolds of two commuting and two anticommuting independent variables by means of the hierarchy of Hamiltonian Lax type flows in the space dual to the central extension of its parameterized version.

On the introduced Lie algebra the non-standard splitting generates some $\mathcal{R}$-operator which is used to obtain a new $\mathcal{R}$-deformed commutator on the corresponding central extension. The mentioned Hamiltonian Lax type flows are determined by the Lie-Poisson bracket associated with $\mathcal{R}$-deformed commutator on the central extension and Casimir functionals as Hamiltonians. The rational factorization problem for these Hamiltonian flows are also studied.

By means of the Backlund mapping given by the rational factorization, one shows that the system of Hamiltonian Lax type flows for a pair of different elements of the parameterized Lie algebra of matrix super-integro-differential operators with two anticommuting variables, which are related by the generalized gauge transformation, is equivalent to the system of two evolution equations for the polynomials in two superderivatives, rationally factorizing these elements, and proves the Hamiltonicity of the latter system. A new method for finding the Poisson operator that generates the corresponding Hamiltonian representation in an explicit form is proposed.

The rational factorization method substantiated in this report can be applied to construct new integrable hierarchies of nonlinear dynamical systems on supermatrix-valued supermanifolds of two commuting and two anticommuting independent variables and infinite sequences of their conservation law gradients.

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Zero-Divisors in Commutative Rings and Their Graph-Theoretic Applications

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Zero-divisors play an important role in the study of commutative rings, since they reveal essential information about the internal structure of a ring, its ideals, and its decomposition properties. In this talk, we study zero-divisors in commutative rings through the language of graph theory. More precisely, to a commutative ring R with identity, one associates the zero-divisor graph $\Gamma(R)$, whose vertices are the nonzero zero-divisors of R, and where two distinct vertices x and y are adjacent whenever

$$xy = 0.$$

This construction translates an algebraic relation into a combinatorial object, allowing ring-theoretic properties to be investigated by means of graph-theoretic tools.

The aim of the talk is to present the main algebraic and graph-theoretic ideas behind this construction. We first recall the basic notions of commutative rings, zero-divisors, nilpotent elements, idempotent elements, ideals, and annihilators. We then explain how the structure of zero-divisors influences the shape of the associated graph. In particular, we discuss graph invariants such as connectedness, diameter, girth, clique number, independence number, chromatic number, and planarity. These invariants provide useful information about the way zero-divisors interact inside the ring.

Special attention is given to finite commutative rings, where the zero-divisor graph gives a concrete and visual representation of algebraic phenomena. Examples such as $\mathbb{Z}/n\mathbb{Z}$, especially for composite integers n, illustrate how the factorization of n affects the set of zero-divisors and the corresponding graph. The talk also highlights the connection between zero-divisor graphs and ideal structures, showing how adjacency relations may reflect annihilation, ideal intersection, and decomposition behavior.

Finally, we indicate some possible applications and perspectives of zero-divisor graphs in computational algebra, coding theory, cryptography, and network-type models. The general purpose is to show that zero-divisors, far from being merely algebraic obstacles, can serve as a bridge between commutative algebra and graph theory, offering both structural insight and useful combinatorial interpretations.

A gauge theoretical generalisation of Bryant's correspondence

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A classical theorem in the theory of minimal surfaces establishes a correspondence between minimal surfaces in $\mathbb{R}^n$ and null holomorphic curves in $\mathbb{C}^n$. A hyperbolic version of this correspondence is due to Bryant: null holomorphic curves in $\mathrm{SL}(2,\mathbb{C})$ correspond to CMC-1 surfaces in the hyperbolic space $\mathbb{H}^3$. We also have a relativistic Bryant type correspondence: CMC-1 immersions in the hyperbolic space are replaced by space-like CMC-1 immersion in the de Sitter space.

We prove a mutual generalisation of all these results: let H be a real Lie group, $\pi : P \to M$ a principal H-bundle, A a connection on P and $\alpha \in A^1_{\mathrm{Ad}}(P, \mathfrak{h})$ a tensorial 1-form of type Ad which induces isomorphisms $A_\xi \to \mathfrak{h}$.

Such a pair (α, A) defines an almost complex structure J^α_A on P, which is integrable if and only (α, A) solves a gauge-invariant non-linear first order differential system. A non-degenerate symmetric Ad_H-invariant bilinear form g on $\mathfrak{h}$ defines pseudo-Riemannian metrics g^α_M, $\mathfrak{g}^\alpha_A$ on M, respectively P, and a non-degenerate bilinear form $\omega^{\alpha,g}_A : T_P \times_P T_P \to \mathbb{C}$ which is holomorphic when J^α_A is integrable.

Assuming that this is the case, we have a Bryant type correspondence between space-like, $\omega^{\alpha,g}_A$-isotropic holomorphic immersions $Y \to P$ and space-like conformal immersions $Y \to (M, g^\alpha_M)$ whose mean curvature vector field is given by a simple explicit formula.

In particular, one obtains such a correspondence for any principal bundle of the form $G \to G/H$, where G is a complex Lie group, and H is a real form of G endowed with a non-degenerate, Ad_H-invariant, symmetric bilinear form g on its Lie-algebra $\mathfrak{h}$.

A stochastic Poisson coalgebra method and their applications

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This talk provides a practical approach to stochastic Lie systems, i.e. stochastic differential equations whose general solutions can be written as a function depending only on a generic family of particular solutions and some constants related to initial conditions. We correct the stochastic Lie theorem characterising stochastic Lie systems, it retains its classical form in the Stratonovich approach. New generalisations of stochastic Lie systems, like the so-called stochastic foliated Lie systems, are introduced. Subsequently, I focus on stochastic Lie systems that are Hamiltonian systems relative to different geometric structures. Special attention is paid to the symplectic case. I study their stability properties and lay the foundations of a stochastic energy-momentum method. A stochastic Poisson coalgebra method is developed to derive superposition rules for Hamiltonian stochastic Lie systems. The present results complement previous approaches through the use of stochastic differential equations, rather than deterministic equations intended to reproduce certain features of stochastic models.

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A Quantum Ground Operator in Field Theory

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The ground operator is the operator that underlies every action of a field and that preserves the energy state of a system, maintaining the law of conservation of energy of the dynamical system given in the Lagrangian.and giving it a direction in space-time. Said operator will be a fundamental part in the system transformations in field theory and to define the field intentionality as the integral

$$\jmath_{\mathfrak{Q}_x}(x(s)) = \int_{\mathfrak{Q}_x}\left\{\int_{\mathcal{M}} L\left(\theta\left(\pi^{-1}\left(\sigma\left(\rho^{-1}(x)\right)\right)\right)\right)\right\}$$

An immediate application is in nanotechnology.

Proposition 1. *The following diagram is commutative:*

$$\begin{array}{ccccc}
T\mathbb{M} & \xrightarrow{O_c} & T\mathbb{M}^* & \xrightarrow{O_c(\upsilon)} & \Omega^2(\mathbb{M}) \\
\Im\downarrow & \nwarrow & \downarrow \pi & \searrow\cong & \downarrow O_c(\upsilon)w, \\
\mathbb{R} & \xrightarrow{\Gamma} & T\mathbb{M} & \xrightarrow{L} & \Omega^1(\mathbb{M})
\end{array}$$

Theorem 2. ***(F. Bulnes)*** *The energy wrapping (spectrum) is characterized by the fields related by diffeomorphism $\boldsymbol{C}_{\_*}(\Omega(x)) \to \mathcal{W}(H)$, whose space of paths going from $\gamma(x)$ to $\phi(x)$, foreseen in* [2]. *Then the ramification of field in this case is the connection to the operador $O_c : TM \to T^*M$*

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Admissible transformations and Lie symmetries of linear Schrödinger equations with complex-valued time-independent potentials

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The study of Lie symmetries for Schrödinger equations was started in the 1970s with the linear case, and it has been continued up to now for more complicated classes of linear Schrödinger equations, see [1, 2, 3, 4] and references therein. Admissible transformations and Lie symmetries of $(1+n)$-dimensional linear Schrödinger equations with (in general) time-dependent complex-valued potentials, which are of the general form

$$i\psi_t + \psi_{aa} + V(t,x)\psi = 0, \tag{1}$$

were studied in [1] for $n=1$ and in [2] for arbitrary $n \in \mathbb{N}$. Here t and $x = (x_1, \dots, x_n)$ are the real independent variables, ψ is the complex dependent variable and V is an arbitrary smooth complex-valued potential depending on t and x. Here and in what follows the indices a runs from 1 to n, and we assume summation over repeated indices. In [1], the notion of uniformly semi-normalized classes of differential equations was introduced and the algebraic method of group classification for such classes was suggested. The results was developed in [2]. The equivalence groupoid $\mathcal{G}^{\sim}_{\mathcal{F}}$ of the class $\mathcal{F}$ of the above equations was computed, and it was thus shown that this class is uniformly semi-normalized with respect to the linear superposition of solutions. Hence, the group classification of $\mathcal{F}$ reduces to the classification of specific low-dimensional subalgebras of the associated equivalence algebra. The group classification problem for the class $\mathcal{F}$ was completely solved in [1] and in [2] for space dimensions one and two, respectively.

In the present work, we consider the subclass $\mathcal{F}'$ of the class $\mathcal{F}$ that consist of the equations of the form (1) with time-independent complex-valued potentials. Based on the description of $\mathcal{G}^{\sim}_{\mathcal{F}}$ in [2], we construct the equivalence group $G^{\sim}_{\mathcal{F}'}$ of this subclass and describe its equivalence groupoid via classifying the admissible transformations within this subclass. This gives the first example in the literature on applying the method of furcate splitting to classifying admissible transformations within a non-normalized class of differential equations.

Theorem 1. *The equivalence group $G^\sim_{\mathcal{F}'}$ of the class $\mathcal{F}'$ consists of the point transformations in the space with the coordinates (t,x,ψ,ψ^*,V,V^*) whose (t,x,V)-components are of the form*

$$\tilde t=\lambda_1 t+\lambda_0,\quad \tilde x=\varepsilon|\lambda_1|^{1/2}x+\nu,\quad \tilde\psi=\mathrm{e}^{i\lambda_3 t+i\lambda_2+\lambda_5 t+\lambda_4}\hat\psi,\quad \tilde V=\frac{\hat V}{|\lambda_1|}+\frac{\lambda_3-i\lambda_5}{\lambda_1},$$

where λ_0, ..., λ_5 and ν are real constants with $\lambda_1\neq 0$, and $\varepsilon:=\pm1$.

Theorem 2. *A generating (up to the $G^\sim_{\mathcal{F}'}$-equivalence and the linear superposition of solutions) set of admissible transformations $(V,\Phi,\tilde V)$ for $\mathcal{F}'$ is the union of the families*

$$\mathcal{T}_{1F}:=\left(x_n^{-2}F([x_1:\ldots:x_n])-x_ax_a,\ \Phi_1,\ \tilde x_n^{-2}F([\tilde x_1:\ldots:\tilde x_n])\right),$$
$$\Phi_1\colon\quad \tilde t=\frac12\tan 2t,\quad \tilde x_a=\frac{x_a}{\cos 2t},\quad \tilde\psi=|\cos 2t|^{n/2}\mathrm{e}^{i\tan(2t)|\boldsymbol{x}|^2/2}\psi,$$

$$\mathcal{T}_{2F}:=\left(x_n^{-2}F([x_1:\ldots:x_n])+x_ax_a,\ \Phi_2,\ \tilde x_n^{-2}F([\tilde x_1:\ldots:\tilde x_n])\right),$$
$$\Phi_2\colon\quad \tilde t=\frac14\mathrm{e}^{4t},\quad \tilde x_a=\mathrm{e}^{2t}x_a,\quad \tilde\psi=\mathrm{e}^{i|\boldsymbol{x}|^2/2-nt}\psi,$$

$$\mathcal{T}_{3\alpha U}:=\big(U(x_2,\ldots,x_n)+i\alpha x_1+x_1,\ \Phi_{3\alpha},\ U(\tilde x_2,\ldots,\tilde x_n)+i\alpha\tilde x_1\big),$$
$$\Phi_{3\alpha}\colon\quad \tilde t=t,\quad \tilde x_1=x_1-t^2,\quad \tilde x_a=x_a,\ a\neq1,\quad \tilde\psi=\mathrm{e}^{-itx_1+(i+\alpha)t^3/3}\psi,$$

$$\mathcal{T}_{4\alpha U\kappa}:=\big(U(x_2,\ldots,x_n)+i\alpha x_1-x_1^2,\ \Phi_{4\alpha\kappa},\ U(\tilde x_2,\ldots,\tilde x_n)+i\alpha\tilde x_1-\tilde x_1^2\big),$$
$$\Phi_{4\alpha\kappa}\colon\quad \tilde t=t,\quad \tilde x_1=x_1+2\kappa\cos 2t,\quad \tilde x_a=x_a,\ a\neq1,\quad \tilde\psi=\mathrm{e}^{\kappa\sin t\,(-2ix_1-2i\kappa\cos 2t-\alpha)}\psi,$$

$$\mathcal{T}_{5\alpha U\kappa\nu}:=\big(U(x_2,\ldots,x_n)+i\alpha x_1+x_1^2,\ \Phi_{5\alpha\kappa\nu},\ U(\tilde x_2,\ldots,\tilde x_n)+i\alpha\tilde x_1+\tilde x_1^2\big),$$
$$\Phi_{5\alpha\kappa\nu}\colon\quad \tilde t=t,\quad \tilde x_1=x_1+2\kappa\mathrm{e}^{2t}+2\nu\mathrm{e}^{-2t},\quad \tilde x_a=x_a,\ a\neq1,\quad \tilde\psi=\mathrm{e}^{(\kappa\mathrm{e}^{2t}-\nu\mathrm{e}^{-2t})(2i(x_1+\kappa\mathrm{e}^{2t}+\nu\mathrm{e}^{-2t})-\alpha)}\psi,$$

where $[x_k:\ldots:x_n]$ with $k\in\{1,\ldots,n\}$ denotes homogeneous coordinates in the projective space of dimension $n-k$, F is a general sufficiently smooth complex-valued function in this space with $k=1$, U is a general sufficiently smooth complex-valued function of $(x_2,\ldots,x_n)$, $\alpha,\kappa,\nu\in\mathbb{R}$, $\kappa\neq0$ in the fourth family, and $(\kappa,\nu)\neq(0,0)$ in the fifth family. Moreover, in the last two families, $\alpha\neq0$ or $x_2U_2+\cdots+x_nU_n+2U\neq4\epsilon(x_2^2+\cdots+x_n^2)$, where $\epsilon=-1$ and $\epsilon=1$ for the fourth and the fifth families, respectively.

Since a class $\mathcal{F}'$ possesses no normalization properties, we use results of [1, 2] and the above theorems to exhaustively solve the group classification problems for this subclass in space dimensions one and two up to the general point equivalence and up to the equivalences generated by the corresponding equivalence groups. But the group classification problem for $\mathcal{F}'$ and, even more so, for $\mathcal{F}$ in space dimension three are much more cumbersome than their counterparts for $n=1$ and $n=2$. Since the group classification problem for the class $\mathcal{F}$ has not been solved for $n>2$ and, we think, will not be completely solved in the near future, we directly solve the group classification problem for the class $\mathcal{F}'$ with $n=3$ up to the both $G^\sim_{\mathcal{F}'}$- and $\mathcal{G}^\sim_{\mathcal{F}'}$-equivalences using an *original algebraic version of the method of furcate splitting.* Then *we reduce the classification of essential Lie symmetry extensions in the class $\mathcal{F}'$ with $n=3$ to the classification of nonzero subalgebras of the extended Euclidean algebra* $\bar{\mathrm{e}}(3)$. Based on the well-known complete list of nonzero inequivalent (with respect to inner automorphisms) subalgebras of $\bar{\mathrm{e}}(3)$ presented in [5], we use an optimized procedure to construct a complete list of $G^\sim_{\mathcal{F}'}$-inequivalent essential Lie-symmetry extensions within the class $\mathcal{F}'$.

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Про геодезичні відображення слабо рекурентних псевдоріманових просторів

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Псевдоріманові простори, в яких існує тензор $T^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_m}$ такий, що

$$T^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_n, k} = \rho_k T^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_m} \tag{1}$$

називають *T-рекурентними.*

А якщо умови (1) виконуються для тензора Рімана, то такі простори називають *рекурентними.*

Векторні поля $u_k \neq 0$, які задовольняють для ненульових тензорів $T^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_m}$ умові

$$u_k T^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_m} + u_{j_1} T^{i_1 i_2 \dots i_r}_{j_2 k \dots j_m} + u_{j_2} T^{i_1 i_2 \dots i_r}_{k j_1 \dots j_n} = 0 \tag{2}$$

називають *векторними оболонками тензора* $T^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_m}$ відносно індексів k, j_1 та j_2.

Якщо векторна оболонка задається відносно кососиметричної пари індексів тензора Рімана, тобто

$$u_i R^h_{jkl} + u_k R^h_{jli} + u_l R^h_{jik} = 0 \tag{3}$$

то вона називається *векторною оболонкою тензора Рімана.*

Враховуючи диференціальну тотожність Біанки, легко переконатися, що в рекурентних псевдоріманових просторах існує векторна оболонка тензора Рімана. Тому псевдоріманові простори, в яких виконуються умови (3), називають *слабо рекурентними просторами.*

Нехай тензор D^h_{ijk} такий, що

$$D^h_{ijk} = R^h_{ijk} - B(\delta^h_k g_{ij} - \delta^h_j g_{ik}),$$

де δ^h_i — символи Кронекера, R_{ijk} —тензор Рімана, а B — деякий інваріант.

Якщо умові (2) буде задовольнять тензор D^h_{ijk}, тобто

$$u_i D^h_{jkl} + u_k D^h_{jli} + u_l D^h_{jik} = 0,$$

то такі простори будемо називать *D–слабо рекурентними псевдорімановими просторами.*

В роботі доведено, якщо D–слаборекурентний псевдоріманів простір V_n допускає нетривіальні геодезичні відображення, то в ньому виконується принаймні одна з таких умов

$$M_{kj} - \frac{M}{n} g_{kj} = 0 \quad \textit{або} \quad a_{\alpha i} u^\alpha = \stackrel{1}{\tau} u_i,$$

де a_{ij} є тензором з лінійної форми основних рівнянь теорії геодезичних деформацій, $M = M_{\alpha\beta} g^{\alpha\beta}$, $M_{ij} = R_{ij} - B(n-2) g_{ij}$.

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