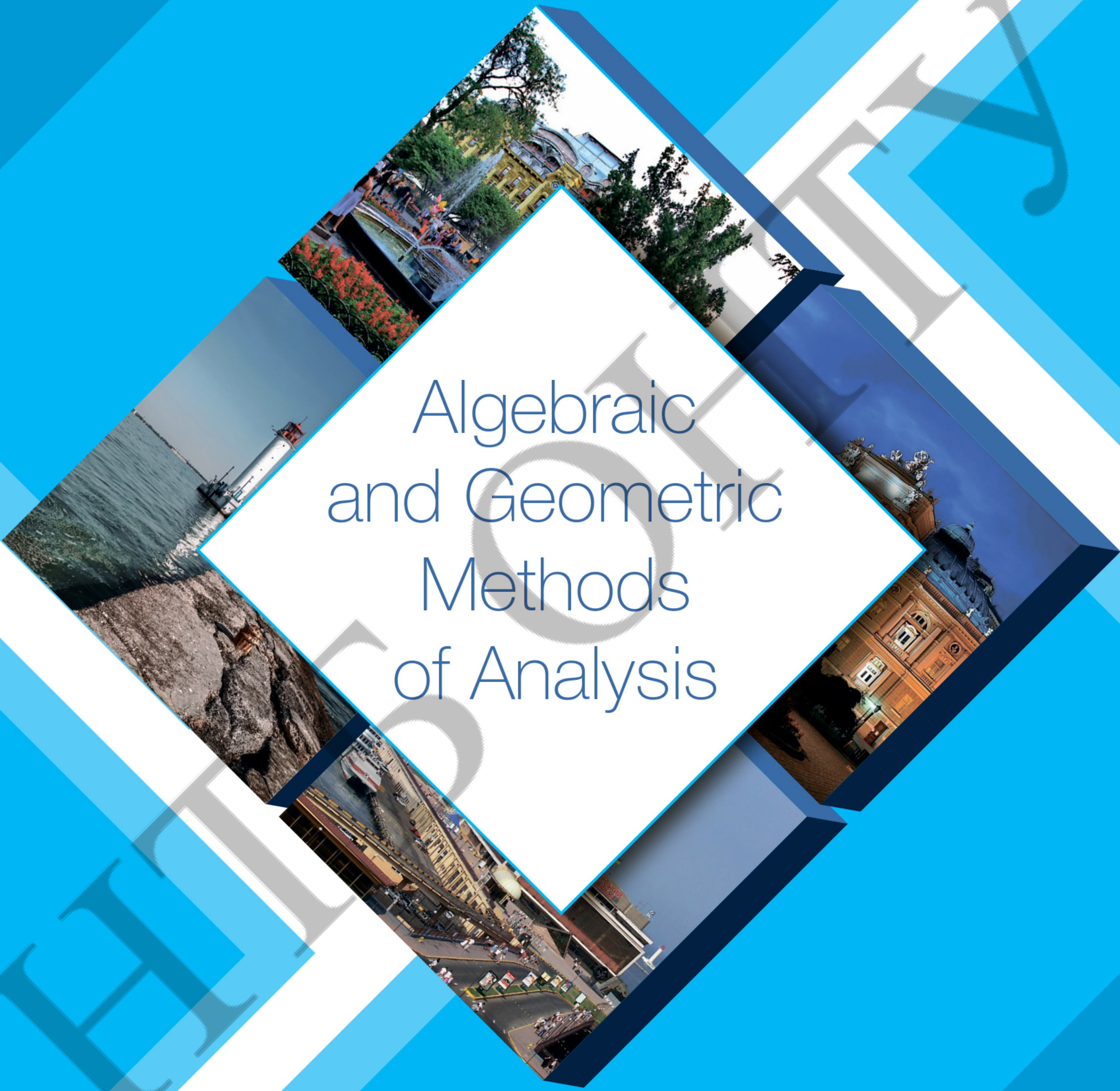
International
Scientific Conference



Algebraic and Geometric Methods of Analysis

27-30 May 2024

Odesa, Ukraine

The purpose of this conference is to bring together researchers in geometry, topology, algebra, analysis and dynamical systems and to provide for them a forum to present their recent work to colleagues from different nationalities. This way we aim to stimulate discussion about the latest findings in geometrical and topological methods in analysis and to increase international collaboration.

The conference continues the traditional annual conference «Geometry in Odesa» holding from 2004, and hosted by Odesa National University of Technology (Odesa National Academy of Food Technologies till 2021). From 2017 the conference was renamed to «Algebraic and geometric methods of analysis» (AGMA).

The Conference languages: Ukrainian and English.

List of topics

- Algebraic methods in geometry
- Differential geometry in the large
- Geometry and topology of differentiable manifolds
- General and algebraic topology
- Dynamical systems and their applications
- Geometric and topological methods in natural sciences
- Geometric problems in mathematical analysis

Organizers

- Ministry of Education and Science of Ukraine
- Odesa National University of Technology, Ukraine
- Institute of Mathematics of the National Academy of Sciences of Ukraine
- Taras Shevchenko National University of Kyiv
- Kyiv Mathematical Society

Scientific Committee

- **Vladimir Balan** (*Bucharest, Romania*)
- **Taras Banakh** (*Lviv, Ukraine*)
- **Dmytro Bolotov** (*Kharkiv, Ukraine*)
- **Vyacheslav Boyko** (*Kyiv, Ukraine*)
- **Yulia Fedchenko** (*Odesa, Ukraine*)
- **Oleg Gutik** (*Lviv, Ukraine*)
- **Olena Karlova** (*Chernivtsi, Ukraine*)
- **Volodymyr Kiosak** (*Odesa, Ukraine*)
- **Nadiia Konovenko** (*Odesa, Ukraine*)
- **Volodymyr Lyubashenko** (*Kyiv, Ukraine*)
- **Sergiy Maksymenko** (*Kyiv, Ukraine*)
- **Koji Matsumoto** (*Yamagata, Japan*)
- **Piotr Mormul** (*Warsaw, Poland*)
- **Maryna Nesterenko** (*Kyiv, Ukraine*)
- **Roman Popovych** (*Kyiv, Ukraine*)
- **Alexandr Prishlyak** (*Kyiv, Ukraine*)
- **Aleksandr Savchenko** (*Kherson, Ukraine*)

Organizing commitee

- **Nadiia Konovenko** (*Odesa, Ukraine*)
- **Yuliya Fedchenko** (*Odesa, Ukraine*)
- **Mykola Lysynskiy** (*Kyiv, Ukraine*)
- **Bohdan Mazhar** (*Kyiv, Ukraine*)
- **Sergiy Maksymenko** (*Kyiv, Ukraine*)
- **Alexandr Prishlyak** (*Kyiv, Ukraine*)

We also proved geodesical convexity of 1-foci ball, which is a hyperbolic ball, with geometrical methods.

References

[1] Constantin Udrişte (auth.). *Convex Functions and Optimization Methods on Riemannian Manifolds.* Mathematics and Its Applications 297. Springer Netherlands, 1 edition, 1994.
[2] Werner Ballmann. *Lectures on Spaces of Nonpositive Curvature.* Birkhäuser Basel, 1995.
[3] John F Barrett. The Hyperbolic Theory of Special Relativity, 2011.
[4] Alan F. Beardon. *The Geometry of Discrete Groups.* Springer New York, 1983.
[5] R. Engelking. *General Topology.* Sigma series in pure mathematics. Heldermann, 1989.
[6] O. P. Ferreira and P. R. Oliveira. Subgradient Algorithm on Riemannian Manifolds. *Journal of Optimization Theory and Applications*, 97(1):93–104, April 1998.
[7] Orizon Pereira Ferreira, Sándor Zoltán Németh, and Jinzhen Zhu. Convexity of sets and quadratic functions on the hyperbolic space, 2021.
[8] Martin Von Gagern. Creation of Hyperbolic Ornaments — Algorithmic and Interactive Methods, 2014.
[9] Eduardo Gallego and Gil Solanes. Perimeter, Diameter and Area of Convex Sets in the Hyperbolic Plane. *Journal of the London Mathematical Society*, 64(1):161–178, August 2001.
[10] V. Guillemin and A. Pollack. *Differential Topology.* Mathematics Series. Prentice-Hall, 1974.
[11] Martin Himmel. More convex functions by Artin's method, 2014.
[12] John M. Lee. *Riemannian Manifolds.* Springer New York, 1997.
[13] Wancang Ma and David Minda. Hyperbolically convex functions. *Annales Polonici Mathematici*, 60(1):81–100, 1994.
[14] William Ma and David Minda. Hyperbolically convex functions II. *Annales Polonici Mathematici*, 71(3):273–285, 1999.
[15] Bojan Mohar. *Drawing Graphs in the Hyperbolic Plane*, page 127–136. Springer Berlin Heidelberg, 1999.
[16] Junpei Sekino. n-ellipses and the minimum distance sum problem. *Amer. Math. Monthly*, 106(3):193–202, 1999.
[17] Xiao-Ming Zhang and Yu-Dong Wu. Geometrically convex functions and solution of a question. *RGMIA Res. Rep. Coll.*, 7(4), 2004.

A retraction from the space of pseudometrics to the space of ultrapseudometrics

Volodymyr Penhryn
(Vasyl Stefanyk Precarpathian National University, Shevchenka 57, Ivano-Frankivsk, Ukraine)
E-mail: volodymyr.penhryn.22@pnu.edu.ua

Oleh Nykyforchyn
(Vasyl Stefanyk Precarpathian National University, Shevchenka 57, Ivano-Frankivsk, Ukraine)
E-mail: oleh.nykyforchyn@pnu.edu.ua

Definition 1. A **pseudometric** on a set X is a function $d : X \times X \to \mathbb{R}$ that satisfies the following properties for all $x, y, z \in X$:

(1) **Non-negativity**: $d(x, y) \geq 0$.
(2) **Identity of indiscernibles**: $d(x, x) = 0$. (However, it is not required that $d(x, y) = 0$ implies $x = y$, which differentiates a pseudometric from a metric.)
(3) **Symmetry**: $d(x, y) = d(y, x)$.
(4) **Triangle inequality**: $d(x, z) \leq d(x, y) + d(y, z)$.

An **ultrapseudometric** is a type of distance function defined on a set that generalizes the notion of a metric, incorporating properties specific to ultrametrics and pseudometrics. Formally:

Definition 2. An ultrapseudometric d on a set X is a function $d : X \times X \to \mathbb{R}$ that satisfies the above properties of **non-negativity, identity of indiscernibles, and symmetry**, but the **triangle inequality** is satisfied in a stronger form:

(4) **Strong triangle inequality (Ultrametric inequality)**[1] : for all $x, y, z \in X$

$$d(x,z) \leq \max\{d(x,y), d(y,z)\}$$

We denote with $\mathcal{P}\int(X)$ the set of all pseudometrics on a fixed set X, and $\mathcal{UP}\int(X)$ is its subset consisting of all ultrapseudometrics on X.

Theorem 3. *There is a non-expanding w.r.t. the uniform convergence metric retraction* $\mathcal{P}\int(X) \to \mathcal{UP}\int(X)$

We rely on the following lemmas.

Lemma 4. *For each pseudometric* $d : X\times X \to \mathbb{R}$ *and a subset* $A \subset X$ *the function* $d_A : X \times X \to \mathbb{R}$ *with the formula*

$$d_A(x,y) = \begin{cases} 0, & x, y \in A \text{ or } x, y \notin A, \\ d(A, X \setminus A), & x \in A, y \notin A \text{ or } x \notin A, y \in A, \end{cases} \quad x, y \in X,$$

is an ultrapseudometric such that $d_A \leq d$.

Proof. Non-negativity, symmetry, and identity of indiscernibles clearly hold. The only not so trivial part is the strong triangle inequality.[2]

- If all three points are in A or all three are not in A: $d_A(x,z) = 0 \leq \max\{0,0\}$.
- If $x, y \in A$ and $z \notin A$ (or vice versa): $d_A(x,y) = 0$, $d_A(y,z) = d(A, X \setminus A)$, $d_A(x,z) = d(A, X \setminus A)$, hence $d_A(x,z) \leq \max\{0, d(A, X \setminus A)\}$.
- If $x \in A$, $y \notin A$, and $z \in A$ (or vice versa): $d_A(x,y) = d(A, X \setminus A)$, $d_A(y,z) = d(A, X \setminus A)$, $d_A(x,z) = 0$, hence $d_A(x,z) \leq \max\{d(A, X \setminus A), d(A, X \setminus A)\}$.

To compare d_A and d:

- If $x, y \in A$ or $x, y \notin A$: $d_A(x,y) = 0 \leq d(x,y)$.
- If $x \in A$ and $y \notin A$ (or vice versa): $d_A(x,y) = d(A, X \setminus A) \leq d(x,y)$.

□

Lemma 5. *For each pseudometric* $d : X \times X \to \mathbb{R}$ *the function* $\bar{d} : X \times X \to \mathbb{R}$ *such that*

$$\bar{d}(x,y) = \sup\{d_A(x,y) \mid A \subset X\}, \quad x, y \in X,$$

is the greatest ultrapseudometric on X *not exceeeding* d.

Proof. (1) **Ultrapseudometric properties of** $\overline{d}$**:**

(a) **Symmetry:**

$$\overline{d}(x,y) = \sup\{d_A(x,y) \mid A \subseteq X\} = \sup\{d_A(y,x) \mid A \subseteq X\} = \overline{d}(y,x)$$

since d_A is symmetric for all $A \subseteq X$.

(b) **Non-negativity and zero distance:**

$$\overline{d}(x,y) \geq 0$$

and

$$\overline{d}(x,x) = \sup\{d_A(x,x) \mid A \subseteq X\} = 0$$

since $d_A(x,x) = 0$ for all $A \subseteq X$.

(c) **Ultrametric inequality:** For all $x, y, z \in X$:
$$\overline{d}(x,z) = \sup\{d_A(x,z) \mid A \subseteq X\}$$
and
$$\overline{d}(x,z) \le \sup\{\max\{d_A(x,y), d_A(y,z)\} \mid A \subseteq X\} \le \max\{\overline{d}(x,y), \overline{d}(y,z)\}$$
since d_A satisfies the ultrametric inequality for all $A \subseteq X$.

(2) **Comparison $\overline{d} \le d$:** For each $A \subseteq X$, we have $d_A \le d$, thus:
$$\overline{d}(x,y) = \sup\{d_A(x,y) \mid A \subseteq X\} \le d(x,y).$$

(3) **Greatest ultrapseudometric not exceeding d:** Suppose there exists another ultrapseudometric d' on X such that $d' \le d$ and $d' \ge \overline{d}$. Then, for any $A \subseteq X$, $d_A \le d'$, hence:
$$\overline{d} = \sup\{d_A \mid A \subseteq X\} \le d'.$$

Therefore, $\overline{d}(x,y) = \sup\{d_A(x,y) \mid A \subseteq X\}$ is the greatest ultrapseudometric on X not exceeding d. □

We will discuss efficient algorithms for calculation of $\bar{d}$ for a given d on a finite set X.

References

[1] de Groot J. Non-archimedean metrics in topology. *Proc. Amer. Math. Soc.*, 7: 948–953, 1956.

[2] Nykorovych S.I., Nykyforchyn O.R., Zagorodnyuk A.V. Approximation Relations on the Posets of Pseudoultrametrics. *Axioms*, 12(5): 438, 2023.

Action of derivations on polynomials and on Jacobian derivations

Oleksandra Kozachok
(Taras Shevchenko National University of Kyiv, 64, Volodymyrska street, 01033 Kyiv, Ukraine)
E-mail: sashulychka@ukr.net

Anatoliy Petravchuk
(Taras Shevchenko National University of Kyiv, 64, Volodymyrska street, 01033 Kyiv, Ukraine)
E-mail: petravchuk@knu.ua

Let $\mathbb{K}$ be an arbitrary field of characteristic zero. Denote by $A := \mathbb{K}[x_1, \dots, x_n]$ the polynomial ring, and by $R := \mathbb{K}(x_1, \dots, x_n)$ the field of rational functions in n variables, respectively. A $\mathbb{K}$-linear map $D : A \to A$ is called a $\mathbb{K}$-derivation on A if $D(fg) = D(f)g + fD(g)$ for any $f, g \in A$. The vector space $W_n(\mathbb{K})$ (over $\mathbb{K}$) of all $\mathbb{K}$-derivation is a Lie algebra with respect to the Lie bracket $[D_1, D_2] = D_1D_2 - D_2D_1$, $D_1, D_2 \in W_n(\mathbb{K})$. Recall that every element $D \in W_n(\mathbb{K})$ can be uniquely written in the form
$$D = f_1\frac{\partial}{\partial x_1} + \cdots + f_n\frac{\partial}{\partial x_n}, f_i \in A.$$
The latter means that $W_n(\mathbb{K})$ is a free module of rank n over A with the free generators $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ (see, for example [3], [4]).

Every element D from $W_n(\mathbb{K})$ acts naturally on polynomials from A and on $W_n(\mathbb{K})$ itself (by multiplication). Recall that a polynomial $f \in A$ is a Darboux polynomial for a derivation $D \in W_n(\mathbb{K})$ if $D(f) = \lambda f$ for some $\lambda \in A$, the polynomial λ is called a cofactor for D. One can consider the Darboux polynomials as "eigenvectors" for the derivation D with polynomial "eigenvalues". These (non-constant) polynomials (if they do exist) play significant role in theory of differential