International
Scientific Conference

Algebraic and Geometric Methods of Analysis

Devoted to 160 anniversary of
Dvytro Grave
(25.08.1863 - 19.12.1939)
Academician of the Ukrainian Academy of Sciences, the first director of the Institute of Mathematics of NAS of Ukraine

May 29 – June 1, 2023
Odesa, Ukraine

List of topics

- Algebraic methods in geometry
- Differential geometry in the large
- Geometry and topology of differentiable manifolds
- General and algebraic topology
- Dynamical systems and their applications
- Geometric and topological methods in natural sciences
- Geometric problems in mathematical analysis

Organizers

- Ministry of Education and Science of Ukraine
- Odesa National University of Technology
- Institute of Mathematics of the National Academy of Sciences of Ukraine
- Taras Shevchenko National University of Kyiv
- Kyiv Mathematical Society

Scientific Committee

- **Bolotov D.** (*Kharkiv, Ukraine*)
- **Bondarenko V.** (*Kyiv, Ukraine*)
- **Boychuk O.** (*Kyiv, Ukraine*)
- **Boyko V.** (*Kyiv, Ukraine*)
- **Cherevko Ye.** (*Odesa, Ukraine*)
- **Dorogovtsev A.** (*Kyiv, Ukraine*)
- **Drozd Yu.** (*Kyiv, Ukraine*)
- **Gerasymenko V.** (*Kyiv, Ukraine*)
- **Fedchenko Yu.** (*Odesa, Ukraine*)
- **Kiosak V.** (*Odesa, Ukraine*)
- **Kochubei A.** (*Kyiv, Ukraine*)
- **Konovenko N.** (*Odesa, Ukraine*)
- **Maksymenko S.** (*Kyiv, Ukraine*)
- **Mikhailets V.** (*Kyiv, Ukraine*)
- **Ostrovskyi V.** (*Kyiv, Ukraine*)
- **Petravchuk A.** (*Kyiv, Ukraine*)
- **Plaksa S.** (*Kyiv, Ukraine*)
- **Portenko M.** (*Kyiv, Ukraine*)
- **Pratsiovytyi M.** (*Kyiv, Ukraine*)
- **Savchenko O.** (*Kherson, Ukraine*)
- **Romanyuk A.** (*Kyiv, Ukraine*)
- **Timokha O.** (*Kyiv, Ukraine*)

Organizing commitee

- **Maksymenko S.** (*Kyiv, Ukraine*)
- **Konovenko N.** (*Odesa, Ukraine*)
- **Fedchenko Yu.** (*Odesa, Ukraine*)
- **Cherevko Ye.** (*Odesa, Ukraine*)
- **Osadchuk Ye.** (*Odesa, Ukraine*)
- **Sergeeva O.** (*Odesa, Ukraine*)

Example 1.9. From Corollary 1.8, there is a $\boldsymbol{gcat}(|\{[\boldsymbol{h}]\}|$ such that $\boldsymbol{gcat}(|\{[\boldsymbol{h}]\}|$ is a minimal covering of the triangle cluster $\boldsymbol{\{\triangle pqr\}} \cap \boldsymbol{p}$, which is a bounded region in Fig. 0.1.

2. Minimal Video Frame Foreground Object Covering

Delaunay triangulations[2] represent pieces of a continuous space in form of triangles with edges attached to selected vertices[3].

Theorem 2.1. *Let $\mathcal{TC}$ be a Delaunay triangle cluster with k triangles minimally covering a planar bounded region $E \in 2^{\mathbb{R}^2}$. Then $\boldsymbol{gcat(\mathcal{TC}) = k}$.*

Example 2.2. With restrictions on the selection of vertices (e.g., centroids), we obtain a minimal cluster $\mathcal{TC}$ of k triangles covering a bus, which is a bounded region in a Delauany triangulation of the video frame foreground in Fig. 1.2. Hence, from Theorem 2.1, $\boldsymbol{gcat(\mathcal{TC}) = k}$.

Fixed point theorem for mappings contracting perimeters of triangles and its generalizations

Evgeniy Petrov
(Institute of Applied Mathematics and Mechanics of the NAS of Ukraine, Slovyansk, Ukraine)
E-mail: eugeniy.petrov@gmail.com

Ruslan Salimov
(Institute of Mathematics of the NAS of Ukraine, Kiev, Ukraine)
E-mail: ruslan.salimov1@gmail.com

We establish two generalizations of the fixed point theorem for mappings contracting perimeters of triangles. In the first case we consider these mappings in semimetric spaces with triangle functions introduced by M. Bessenyei and Z. Páles. Such approach allows us to obtain corollaries for different types of semimetric spaces. In the second case we establish the fixed point theorem in ordinary metric spaces for more general class of mappings than mappings contractive perimeters of triangles.

Let X be a nonempty set. Recall that a mapping $d\colon X\times X \to \mathbb{R}^+$, $\mathbb{R}^+ = [0,\infty)$ is a *metric* if for all $x, y, z \in X$ the following axioms hold: (i) $(d(x, y) = 0) \Leftrightarrow (x = y)$; (ii) $d(x, y) = d(y, x)$; (iii) $d(x, y) \leqslant d(x, z) + d(z, y)$. The pair (X, d) is called a *metric space*. If only axioms (i) and (ii) hold then d is called a *semimetric*. A pair (X, d), where d is a semimetric on X, is called a *semimetric space*.

In 2017 M. Bessenyei and Z. Páles [1] introduced a definition of a triangle function $\Phi\colon \overline{\mathbb{R}}^2_+ \to \overline{\mathbb{R}}^+$ for a semimetric d. We use this definition in a slightly different form restricting the domain and the range of Φ by $\mathbb{R}^2_+$ and $\mathbb{R}^+$, respectively.

[2] B. Delaunay, Sur la sphère vide. a la mémoire de georges voronoï, Izvestia Akad. Nauk SSSR, Otdelenie Matematicheskii i Estestvennyka Nauk 7 (1934), 793–800.

[3] J.F. Peters, Proximal Voronoï regions, convex polygons, & Leader uniform topology, Advances in Math.: Sci. J. 4 (2015), no. 1, 1–5.

Definition 1. Consider a semimetric space (X,d). We say that $\Phi\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a *triangle function* for d if Φ is symmetric and monotone increasing in both of its arguments, satisfies $\Phi(0,0)=0$ and, for all $x,y,z \in X$, the generalized triangle inequality

$$d(x,y) \leqslant \Phi(d(x,z), d(z,y))$$

holds.

Definition 2. Let (X,d) be a semimetric space with $|X| \geqslant 3$. We shall say that $T\colon X \to X$ is a *mapping contracting perimeters of triangles* on X if there exists $\alpha \in [0,1)$ such that the inequality

$$d(Tx,Ty) + d(Ty,Tz) + d(Tx,Tz) \leqslant \alpha(d(x,y) + d(y,z) + d(x,z)) \tag{1}$$

holds for all three pairwise distinct points $x,y,z \in X$.

Note that the requirement for $x,y,z \in X$ to be pairwise distinct is essential. One can see that otherwise this definition is equivalent to the definition of contraction mapping.

Theorem 3. *Let (X,d), $|X| \geqslant 3$, be a complete semimetric space with the triangle function Φ satisfying the following three conditions:*

1) The inequality

$$\Phi(k\xi, k\eta) \leqslant k\Phi(\xi,\eta)$$

holds for all $k,\xi,\eta \in \mathbb{R}^+$.

2) For every $0 \leqslant \alpha < 1$ there exists $C(\alpha) > 0$ such that for every $p \in \mathbb{N}^+$ the inequality

$$\Phi(1, \Phi(\alpha, \Phi(\alpha^2, ..., \Phi(\alpha^{p-1}, \alpha^p)))) \leqslant C(\alpha)$$

holds.

3) Φ is continuous at $(0,0)$.

Let the mapping $T\colon X \to X$ satisfy the following two conditions:

(i) $T(T(x)) \neq x$ for all $x \in X$ such that $Tx \neq x$.

(ii) T is a mapping contracting perimeters of triangles on X.

Then T has a fixed point. The number of fixed points is at most two.

Corollary 4. *Theorem 3 holds for semimetric spaces with power triangle functions $\Phi(x,y) = (x^q + y^q)^{\frac{1}{q}}$ if $q > 0$.*

If the usual triangle inequality is replaced by $d(x,y) \leqslant K(d(x,z) + d(z,y))$, $K \geqslant 1$, then (X,d) is called a *b-metric space*. The definition of a b-metric space was introduced by Czerwik [2].

Corollary 5. *Theorem 3 holds for b-metric spaces if $\alpha K < 1$, where α is the coefficient in (1).*

Definition 6. Let (X,d) be a metric space with $|X| \geqslant 3$ and let functions $F, G\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be such that for all $\xi,\eta,\zeta \in \mathbb{R}^+$ the following conditions hold:

$$F(\eta,\xi,\zeta) = F(\xi,\eta,\zeta) = F(\xi,\zeta,\eta),$$
$$G(\eta,\xi,\zeta) = G(\xi,\eta,\zeta) = G(\xi,\zeta,\eta),$$
$$G(\xi,\eta,\zeta) \geqslant \xi,$$

$$F(\xi, \eta, \zeta) \geqslant G(\xi, \eta, \zeta),$$
$$G(0,0,0) = 0 \text{ and } G \text{ is continuous at } (0,0,0).$$

We shall say that $T\colon X \to X$ is an (F, G)-*contracting mapping* on X if there exists $\alpha \in [0, 1)$ such that the inequality

$$F(d(Tx, Ty), d(Ty, Tz), d(Tx, Tz)) \leqslant \alpha\, G(d(x, y), d(y, z), d(x, z))$$

holds for all three pairwise distinct points $x, y, z \in X$.

Theorem 7. *Let (X, d), $|X| \geqslant 3$, be a complete metric space and let $T\colon X \to X$ be a mapping satisfying the following two conditions:*

(i) $T(T(x)) \neq x$ for all $x \in X$ such that $Tx \neq x$.

(ii) T is an (F, G)-contracting mapping on X.

Then T has a fixed point. The number of fixed points is at most two.

If in Theorem 3 we set $\Phi(x, y) = x + y$ or in Theorem 7 we set $F(\xi, \eta, \zeta) = G(\xi, \eta, \zeta) = \xi + \eta + \zeta$, then we get the following.

Corollary 8. *Let (X, d), $|X| \geqslant 3$, be a complete metric space and let the mapping $T\colon X \to X$ satisfy the following two conditions:*

(i) $T(T(x)) \neq x$ for all $x \in X$ such that $Tx \neq x$.

(ii) T is a mapping contracting perimeters of triangles on X.

Then T has a fixed point. The number of fixed points is at most two.

References

[1] M. Bessenyei and Z. Páles. A contraction principle in semimetric spaces. *J. Nonlinear Convex Anal.*, 18(3): 515–524, 2017.

[2] S. Czerwik. Nonlinear set-valued contraction mappings in b-metric spaces. *Atti Semin. Mat. Fis. Univ. Modena*, 46(2): 263–276, 1998.

Structure of codimensional one flows on the 2-sphere with holes

Alexandr Prishlyak

(Taras Shevchenko National University of Kyiv)

E-mail: prishlyak@yahoo.com

First, we consider gradient vector fields on a sphere. Since the function increases along each trajectory, the field has no cycles and polycycles. In general position, a typical gradient field is a Morse field (Morse-Smale field without closed trajectories). In typical one-parameter families of gradient vector fields, two types of bifurcations are possible: saddle-node and saddle connection. The corresponding vector fields at the time of the bifurcation are fields of codimension one. In our case, they completely determine the topological type of the bifurcation. To classify Morse fields, a cell complex (diagram) is often used, in which cells of dimension n are stable manifolds of singular points with Morse index equal to n. We apply this approach to the classification of vector fields of codimension one.

Without loss of generality, we assume that under bifurcation (as the parameter increases), the number of singular points does not increase. The saddle-node bifurcation is defined by a