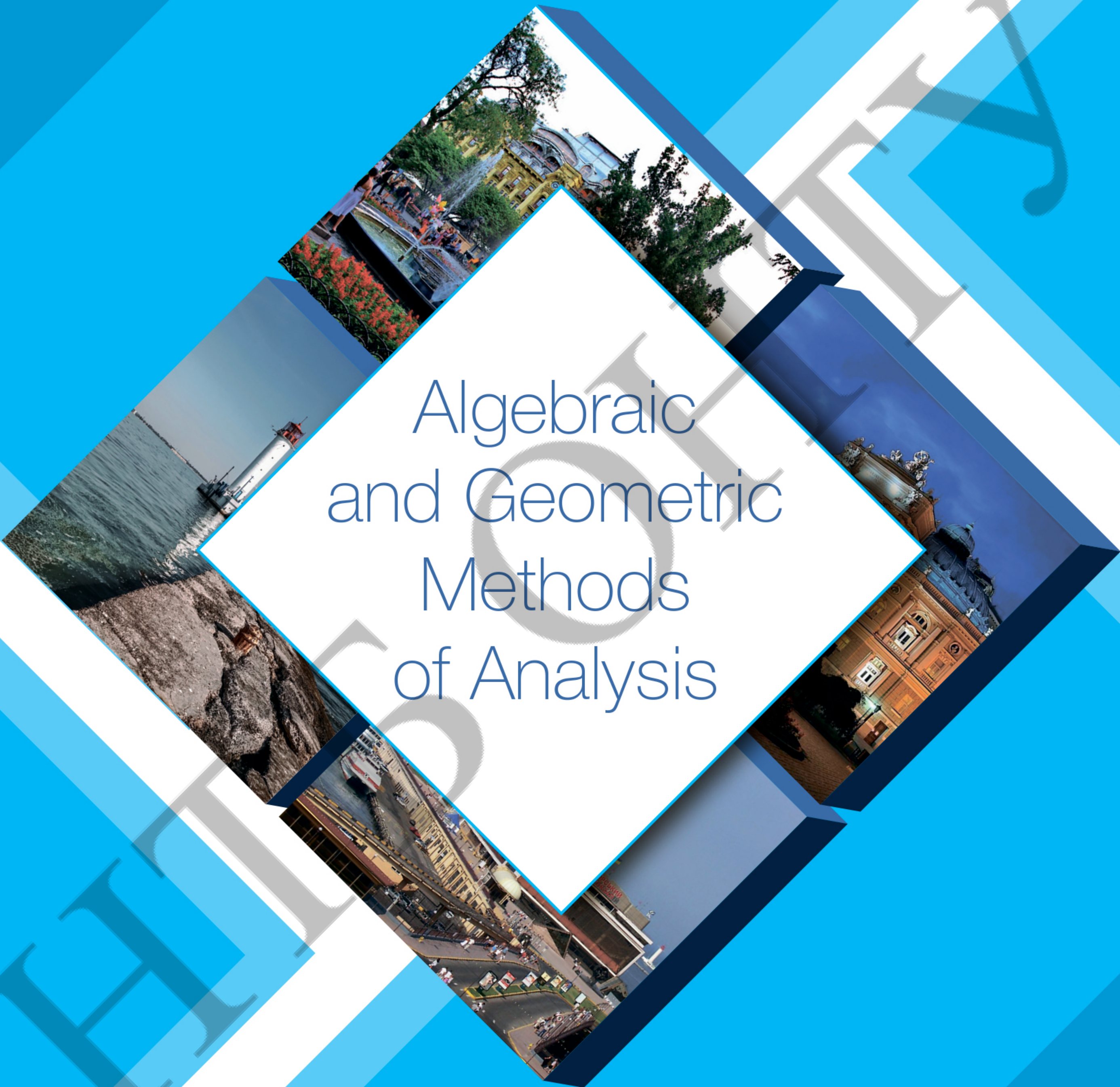
International
Scientific Conference



Algebraic and Geometric Methods of Analysis

27-30 May 2024

Odesa, Ukraine

The purpose of this conference is to bring together researchers in geometry, topology, algebra, analysis and dynamical systems and to provide for them a forum to present their recent work to colleagues from different nationalities. This way we aim to stimulate discussion about the latest findings in geometrical and topological methods in analysis and to increase international collaboration.

The conference continues the traditional annual conference «Geometry in Odesa» holding from 2004, and hosted by Odesa National University of Technology (Odesa National Academy of Food Technologies till 2021). From 2017 the conference was renamed to «Algebraic and geometric methods of analysis» (AGMA).

The Conference languages: Ukrainian and English.

List of topics

- Algebraic methods in geometry
- Differential geometry in the large
- Geometry and topology of differentiable manifolds
- General and algebraic topology
- Dynamical systems and their applications
- Geometric and topological methods in natural sciences
- Geometric problems in mathematical analysis

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Stability of vertical minimal surfaces in three-dimensional sub-Riemannian manifolds

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A *sub-Riemannian manifold* is a smooth manifold M together with a completely non-integrable smooth distribution $\mathcal{H}$ on M (it is called a *horizontal distribution*) and a smooth field of Euclidean scalar products $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ on $\mathcal{H}$ (it is called a *sub-Riemannian metric*). In particular, $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ can be constructed as a restriction of some Riemannian metrics $\langle\cdot,\cdot\rangle$ on M to $\mathcal{H}$. Here we will assume that all sub-Riemannian structures are of this form. Let Σ be a smooth oriented surface in a three-dimensional sub-Riemannian manifold M. If N_h is the orthogonal projection of the unit normal field N of Σ (in the Riemannian sense) onto $\mathcal{H}$ and $d\Sigma$ is the Riemannian area form of Σ, then the *sub-Riemannian area* of a domain $D \subset \Sigma$ is defined as $A(D) = \int\limits_D |N_h|\, d\Sigma$. The *normal variation* of the surface Σ defined by a smooth function u is the map $\varphi\colon \Sigma \times I \to M\colon \varphi_s(p) = \exp_p(su(p)N(p))$, where I is an open neighborhood of 0 in $\mathbb{R}$ and $\exp_p$ is the Riemannian exponential map in p. Denote $A(s) = \int\limits_{\Sigma_s} |N_h|\, d\Sigma_s$, where $\Sigma_s = \varphi_s(\Sigma)$. Then $A'(0)$ is called the *first (normal) area variation* defined by φ, and $A''(0)$ is called the *second* one. A surface Σ is called *minimal* if $A'(0) = 0$ for any normal variations with compact support in $\Sigma \setminus \Sigma_0$, where $\Sigma_0 = \{p \in \Sigma \mid N_h(p) = 0\}$ is the *singular set* of Σ. A minimal surface Σ is called *stable* if $A''(0) \geq 0$ for any normal variations with compact support in $\Sigma \setminus \Sigma_0$. We will call a surface Σ in a three-dimensional sub-Riemannian manifold *vertical* if $T_p\Sigma \perp \mathcal{H}_p$ for each $p \in \Sigma$. In particular, for such surfaces $N_h = N$ and $\Sigma_0 = \varnothing$.

Proposition 1. *Let Σ be an oriented vertical surface in a three-dimensional sub-Riemannian manifold M. Then its first normal area variation defined by a smooth function u with compact support equals $A'(0) = -\int_\Sigma 2Hu\, d\Sigma$, where H is the Riemannian mean curvature of Σ. Thus, Σ is minimal in the sub-Riemannian sense if and only if it is minimal in the Riemannian sense.*

Proposition 2. *Let Σ be an oriented vertical minimal surface in a three-dimensional sub-Riemannian manifold M. Then its second normal area variation defined by a smooth function u with compact support equals*

$$A''(0) = \int_\Sigma -\left(X(u) - \langle \nabla_N X, N\rangle u\right)^2 + |\nabla_\Sigma u|^2 - \left(\operatorname{Ric}(N,N) + |B|^2\right) u^2\, d\Sigma,$$

where ∇ and Ric are the Riemannian connection and the Ricci tensor of M respectively, X is the unit normal vector field of $\mathcal{H}$ (which is tangent to Σ because it is vertical), ∇_Σ and B are the Riemannian gradient and the second fundamental form of Σ respectively. It follows that if Σ is stable in the sub-Riemannian sense, it is also stable in the Riemannian sense.

Let us discuss some examples of such surfaces. In all these examples M is a Lie group, $\mathcal{H}$ and $\langle\cdot,\cdot\rangle$ are left-invariant. In [2] it was shown that a complete connected minimal surface with the empty singular set (in particular, vertical) in the sub-Riemannian three-dimensional Heisenberg group is stable if and only if it is a vertical Euclidean plane. In [3] the authors considered the standard three-dimensional sphere with the horizontal distribution defined by the Hopf field X and showed that complete connected vertical minimal surfaces are Clifford tori. It is well-known that they are not stable in the Riemannian sense, hence also in the sub-Riemannian sense. In [1] we proved that in the solvable Lie group $\widetilde{E(2)}$, which is the universal covering of the proper motions group of the Euclidean plane, with the Euclidean metric and a left-invariant horizontal distribution all complete connected vertical minimal surfaces are Euclidean planes and standard helicoids. We showed that planes are stable in the sub-Riemannian sense, and it is known that helicoids are not stable in the Riemannian sense, hence also in the sub-Riemannian sense.

The three-dimensional Thurston geometry Sol is the space $\mathbb{R}^3$ with coordinates (x, y, z) and with the following orthonormal basis of left-invariant vector fields defined by its solvable Lie group structure:

$$X_1 = \frac{1}{\sqrt{2}}\left(e^{-z}\frac{\partial}{\partial x} + e^{z}\frac{\partial}{\partial y}\right),\ X_2 = \frac{1}{\sqrt{2}}\left(e^{-z}\frac{\partial}{\partial x} - e^{z}\frac{\partial}{\partial y}\right),\ X_3 = \frac{\partial}{\partial z}.$$

Note that $[X_2, X_3] = X_1$, so the left-invariant distribution $\mathcal{H}$ orthogonal to X_1 is completely non-integrable. Let us consider a sub-Riemannian structure on Sol such that $\mathcal{H}$ is horizontal. It then follows from the results of [4] that any complete connected vertical minimal surface in Sol after an isometry becomes either a Euclidean plane $z = C$ or a "helicoid"

$$(s, t) \mapsto \left(\frac{1}{\sqrt{2}}e^{-t}s + C_1, \frac{1}{\sqrt{2}}e^{t}s + C_2, t\right).$$

Using this description, we are able to prove the following.

Proposition 3. *All vertical minimal surfaces in Sol are stable in the sub-Riemannian sense and thus in the Riemannian sense.*

The three-dimensional Thurston geometry $\widetilde{SL(2,\mathbb{R})}$ can be described as the universal covering of the unit tangent bundle of the hyperbolic plane H^2 with the Sasaki metric, that is, the half-space $\{(x, y, z) \in \mathbb{R}^3 \mid y > 0\}$ with the following orthonormal basis of left-invariant vector fields with respect to its simple Lie group structure:

$$X_1 = y\left(-\sin z\frac{\partial}{\partial x} + \cos z\frac{\partial}{\partial y}\right) + \sin z\frac{\partial}{\partial z},\ X_2 = y\left(-\cos z\frac{\partial}{\partial x} - \sin z\frac{\partial}{\partial y}\right) + \cos z\frac{\partial}{\partial z},\ X_3 = \frac{\partial}{\partial z}.$$

In particular, $[X_1, X_2] = -X_3$, so the left-invariant distribution $\mathcal{H}$ orthogonal to X_3 is completely non-integrable. Consider a sub-Riemannian structure on this manifold such that $\mathcal{H}$ is horizontal. We than obtain the following description.

Theorem 4. *Any complete connected vertical minimal surface in $\widetilde{SL(2,\mathbb{R})}$ has either the parameterization $(s, t) \mapsto (C, s, t)$ or $(s, t) \mapsto \left(C_1 + \frac{1}{C_2}\sin C_2 s, -\frac{1}{C_2}\cos C_2 s, t\right)$ and so is a cylinder over a geodesic in H^2. All vertical minimal surfaces in $\widetilde{SL(2,\mathbb{R})}$ are stable in the sub-Riemannian sense and thus in the Riemannian sense.*

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